## 5.1 The Mean Value Theorem

Calculus

Solutions **Practice**

1. Skater Sully is riding a skateboard back and forth on a street that runs north/south. The twice-differentiable function $S$ models Sully's position on the street, measured by how many meters north he is from his starting point, at time $t$, measured in seconds from the start of his ride. The table below gives values of $S(t)$ at selected times $t$.

| $t$ seconds | 0 | 20 | 30 | 60 |
|---|---|---|---|---|
| $S(t)$ meters | 0 | $-5$ | 7 | 40 |

a. For $0 \le t \le 20$, must there be a time $t$ when Sully is 2 meters south of his starting point? Justify your answer.

**$S(t)$ is continuous and $S(0) = 0$ and $S(20) = -5$**

**According to the IVT, there is a value $c$ such that $S(c) = -2$ and $0 \le c \le 20$.**

b. For $30 \le t \le 60$, must there be a time $t$ when Sully's velocity is 1.1 meters per second? Justify your answer.

**$S(t)$ is differentiable and $\frac{40-7}{60-30} = 1.1$**

**Yes. According to the MVT, there must be a value $c$ where $30 \le c \le 60$ and $S'(c) = 1.1$.**

2. A particle is moving along the $x$-axis. The twice-differentiable function $s$ models the particles distance from the origin, measured in centimeters, at time $t$, measured in seconds.

| $t$ seconds | 3 | 10 | 20 | 25 |
|---|---|---|---|---|
| $s(t)$ cm | 5 | $-2$ | $-10$ | 8 |

a. For $20 \leq t \leq 25$, must there be a time $t$ when the particle is at the origin? Justify your answer.

**$s(t)$ is continuous and $s(20) = -10$ and $s(25) = 8$**

**According to the IVT, there is a value $c$ such that $s(c) = 0$ and $20 \leq c \leq 25$.**

b. For $3 \leq t \leq 10$, must there be a time $t$ when the particle's velocity is $-1$ cm per second? Justify your answer.

**$s(t)$ is differentiable and $\frac{-2-5}{10-3} = -1$.**

**Yes. According to the MVT, there must be a value $c$ where $3 \leq c \leq 10$ and $s'(c) = -1$.**

3. A hot air balloon is launched into the air with a human pilot. The twice-differentiable function $h$ models the balloon's height, measured in feet, at time $t$, measured in minutes. The table below gives values of $h(t)$ at selected times $t$.

| $t$ minutes | 0 | 6 | 10 | 40 |
|---|---|---|---|---|
| $h(t)$ feet | 0 | 46 | 35 | 125 |

a. For $6 \leq t \leq 10$, must there be a time $t$ when the balloon is 50 feet in the air? Justify your answer.

**$h(6) = 46$ and $h(10) = 35$. No, the IVT does not guarantee a height of 50.**

b. For $10 \leq t \leq 40$, must there be a time $t$ when the balloon's velocity is 3 feet per second? Justify your answer.

**$h(t)$ is differentiable and $\frac{125-35}{40-10} = \frac{90}{30} = 3$.**

**Yes. According to the MVT, there must be a value $c$ where $10 \leq c \leq 40$ and $h'(c) = 3$.**

**Using the Mean Value Theorem, find where the instantaneous rate of change is equivalent to the average rate of change.**

4. $y = x^2 - 5x + 2$ on $[-4, -2]$

$$\frac{y(-2) - y(-4)}{-2 - -4} = \frac{16-38}{2} = -11$$

$$y' = 2x - 5 = -11$$

$$2x = -6$$

$$\boxed{x = -3}$$

5. $y = \sin 3x$ on $[0, \pi]$

$$\frac{y(\pi) - y(0)}{\pi - 0} = \frac{0-0}{\pi} = 0$$

$$y' = 3\cos(3x) = 0$$

$$\cos(3x) = 0$$

$$3x = \pi/2 \quad 3x = 3\pi/2 \quad 3x = 5\pi/2$$

$$\boxed{x = \pi/6 \quad x = \pi/2 \quad x = 5\pi/6}$$

6. $y = (-5x+15)^{\frac{1}{2}}$ on $[1,3]$

$$\frac{y(3)-y(1)}{3-1} = \frac{0-\sqrt{10}}{2} = -\frac{\sqrt{10}}{2}$$

$$y' = \tfrac{1}{2}(-5x+15)^{-\frac{1}{2}}(-5)$$

$$-\frac{5}{2\sqrt{-5x+15}} = -\frac{\sqrt{10}}{2} \quad \cdot(-2)$$

(-2)

$$\frac{5}{\sqrt{-5x+15}} = \sqrt{10} \leftarrow \text{square both sides}$$

$$\frac{25}{-5x+15} = 10$$

$$25 = -50x + 150$$

$$-125 = -50x$$

$$\boxed{x = 2.5}$$

7. $y = e^x$ on $[0, \ln 2]$

$$\frac{y(\ln 2) - y(0)}{\ln 2 - 0} = \frac{2-1}{\ln 2} = \frac{1}{\ln 2}$$

$$y' = e^x = \frac{1}{\ln 2}$$

$$\boxed{x = \ln\left(\frac{1}{\ln 2}\right) \text{ or } x \approx 0.3665}$$

## 5.1 The Mean Value Theorem

**Test Prep**

Calculator active problem

8. A particle moves along the $x$-axis so that its position at any time $t \geq 0$ is given by $x(t) = t^3 - 3t^2 + t + 1$. For what values of $t$, $0 \leq t \leq 2$, is the particle's instantaneous velocity the same as its average velocity on the closed interval $[0,2]$?

$$\text{Avg. Vel} = \frac{x(2)-x(0)}{2-0} = \frac{-2}{2} = -1$$

$$\text{inst. vel} = 3t^2 - 6t + 1$$

$$3t^2 - 6t + 1 = -1$$

$$3t^2 - 6t + 2 = 0$$

Find zeros

$$\boxed{t \approx 0.4226, \quad t \approx 1.577}$$

No calculator on this problem.

9. The table below gives selected values of a function $f$. The function is twice differentiable with $f''(x) > 0$.

This means $f'$ is increasing

$$\frac{f(5)-f(3)}{5-3} = \frac{1.4}{2} = 0.7$$

$$\frac{f(7)-f(5)}{7-5} = \frac{2.2}{2} = 1.1$$

| $x$ | $f(x)$ |
|---|---|
| 3 | 12.5 |
| 5 | 13.9 |
| 7 | 16.1 |

> 0.7

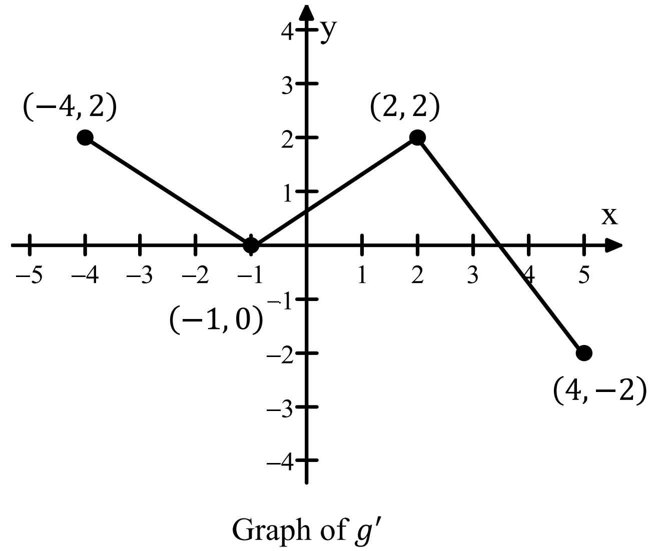
> 1.1

Which of the following could be the value of $f'(5)$?

must be in between 0.7 and 1.1

(A) 0.5 (B) 0.7 (C) 0.9 (D) 1.1 (E) 1.3

10. Let $g$ be a continuous function. The graph of the piecewise-linear function $g'$, the derivative of $g$, is shown above for $-4 \le x \le 4$. Find the average rate of change of $g'(x)$ on the interval $-4 \le x \le 4$. Does the Mean Value Theorem applied on the interval $-4 \le x \le 4$ guarantee a value of $c$, for $-4 < x < 4$, such that $g''(c)$ is equal to this average rate of change? Why or why not?



Graph of $g'$

Avg: $\dfrac{g'(4)-g'(-4)}{4--4} = \dfrac{-2-2}{8} = -\dfrac{1}{2}$

**No, because $g'(x)$ is not differentiable. It has several corners. The MVT only applies if the function is differentiable.**

---

11.

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 1 | 3 | 8 | 2 | 4 |
| 2 | 6 | 3 | 1 | 2 |
| 3 | 5 | $-3$ | 6 | 3 |
| 4 | $-2$ | 6 | 3 | 5 |

The functions $f$ and $g$ are differentiable for all real numbers. The table above gives values of the functions and their first derivatives at selected values of $x$. The function $h$ is given by $h(x) = f(g(x)) + 2$. Must there be a value $c$ for $2 < c < 4$ such that $h'(c) = 1$.

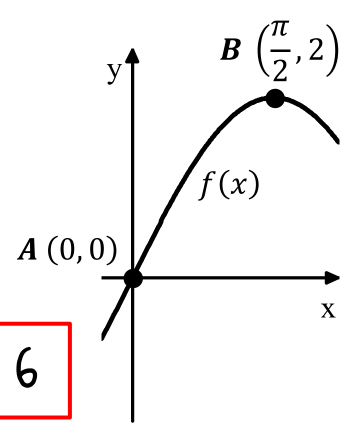
$h(4) = f(g(4)) + 2 = f(3) + 2 = 5 + 2 = 7$

$h(2) = f(g(2)) + 2 = f(1) + 2 = 3 + 2 = 5$

$\dfrac{h(4)-h(2)}{4-2} = \dfrac{7-5}{2} = 1$ ✓ Yes, the MVT proves it.

---

12. **Calculator active problem.** Let $f$ be the function given by $f(x) = 2\sin x$. As shown above, the graph $f$ crosses the origin at point $A$ and point $B$ at the coordinate point $\left(\frac{\pi}{2}, 2\right)$. Find the $x$-coordinate of the point on the graph of $f$, between points $A$ and $B$, at which the line tangent to the graph of $f$ is parallel to line $AB$. Round or truncate to three decimals.




$\dfrac{f(\frac{\pi}{2})-f(0)}{\frac{\pi}{2}-0} = \dfrac{2}{\frac{\pi}{2}} = \dfrac{4}{\pi}$

$f'(x) = 2\cos x$

$2\cos x = \dfrac{4}{\pi}$

graph and find pt. of intersection

$\boxed{x = 0.8806}$

13. A differentiable function $g$ has the property that $g'(x) > 2$ for $1 \le x \le 5$ and $g(4) = 3$. Which of the following could be true?

✓ I. $g(1) = -6$

✗ II. $g(2) = 0$

✗ III. $g(5) = 4$

Slope is steeper than 2.

$\frac{g(4)-g(1)}{4-1} > 2$ ?

$\frac{g(4)-g(2)}{4-2} > 2$ ?

$\frac{g(4)-g(5)}{5-4} > 2$ ?

(A) I only

(B) II only

(C) I and II only

(D) I and III only

(E) II and III only

---

14. **Calculator active problem.** Let $f$ be the function with $f(1) = e$, $f(4) = \frac{1}{e}$, and derivative given by $f'(x) = (x-1)\sin(ex)$. How many values of $x$ in the open interval $(1, 4)$ satisfy the conclusion of the Mean Value Theorem for the function $f$ on the closed interval $[1, 4]$?

$$\frac{f(4)-f(1)}{4-1} = \frac{1/e - e}{3}$$

(A) None

(B) One

(C) Two

(D) More than two

$$\frac{\frac{1}{e} - e}{3} = (x-1)\sin(ex)$$

Graph and count the number of intersections on the interval (1, 4).