8.1 Average Value of a Function
Calculus

Solutions

Practice

Find the average value of each function on the given interval.

1. $f(x) = x^2$ on $[2, 4]$

$$\frac{1}{4-2}\int_2^4 x^2\,dx$$
$$\frac{1}{2}\left[\frac{x^3}{3}\right]\Big|_2^4$$
$$\frac{1}{6}\left[4^3 - 2^3\right]$$
$$\frac{1}{6}\left[64 - 8\right]$$
$$\frac{1}{6}(56) = \boxed{\frac{28}{3}}$$

2. $f(x) = \sin x$ on $[0, \pi]$

$$\frac{1}{\pi - 0}\int_0^\pi \sin x\,dx$$
$$\frac{1}{\pi}\left[-\cos x\right]\Big|_0^\pi$$
$$-\frac{1}{\pi}\left[\cos(\pi) - \cos(0)\right]$$
$$-\frac{1}{\pi}\left[-1 - 1\right]$$
$$\boxed{\frac{2}{\pi}}$$

3. $f(x) = \sqrt{x}$ on $[0, 16]$

$$\frac{1}{16-0}\int_0^{16} x^{\frac{1}{2}}\,dx$$
$$\frac{1}{16}\left[\frac{2}{3}x^{\frac{3}{2}}\right]\Big|_0^{16}$$
$$\frac{1}{24}\left[4^3 - 0\right]$$
$$\frac{1}{6}\cdot 4^2$$
$$\boxed{\frac{8}{3}}$$

4. $f(x) = \frac{1}{x^2}$ on $[-4, -2]$

$$\frac{1}{-2-{-4}}\int_{-4}^{-2} x^{-2}\,dx$$
$$\frac{1}{2}\left[-\frac{1}{x}\right]\Big|_{-4}^{-2}$$
$$-\frac{1}{2}\left[\frac{1}{-2} - \frac{1}{-4}\right]$$
$$-\frac{1}{2}\left[-\frac{2}{4} + \frac{1}{4}\right] = \boxed{\frac{1}{8}}$$

On the given interval, find the x-value where the function is equivalent to the average value on that interval.

5. $f(x) = 2x - 2$ on $[1, 4]$

$$\frac{1}{4-1}\int_1^4 (2x-2)\,dx = 2x-2$$
$$\frac{1}{3}\left[x^2 - 2x\right]\Big|_1^4 = 2x-2$$
$$\left[(16-8) - (1-2)\right] = 6x - 6$$
$$8 + 1 = 6x - 6$$
$$15 = 6x$$
$$\boxed{x = \frac{5}{2}}$$

6. $f(x) = -\frac{x^2}{2}$ on $[0, 3]$

$$\frac{1}{3-0}\int_0^3 \left(-\frac{1}{2}x^2\right)dx = -\frac{x^2}{2}$$
$$-\frac{1}{6}\left[\frac{x^3}{3}\right]\Big|_0^3 = -\frac{x^2}{2}$$
$$-\frac{1}{18}\left[3^3 - 0\right] = -\frac{x^2}{2}$$
$$27 = 9x^2$$
$$3 = x^2$$
$$\boxed{x = \sqrt{3}}$$

Find the average rate of change on the given interval.

7. $f(x) = -(2x-6)^{\frac{2}{3}}$ on $[1,3]$

$$\frac{f(3)-f(1)}{3-1} = \frac{-(6-6)^{2/3} - -(2-6)^{2/3}}{2}$$

$$\frac{0 + (-4)^{2/3}}{2}$$

$$\boxed{\frac{\sqrt[3]{16}}{2}}$$

8. $y = x^3 - 2x^2 + 2$ on $[-1,1]$

$$\frac{(1-2+2)-(-1-2+2)}{1 - -1}$$

$$\frac{(1)-(-1)}{2}$$

$$\boxed{1}$$

Find where the instantaneous rate of change is equivalent to the average rate of change. (MVT)

9. $y = x^2 - 4x + 3$ on $[0,4]$

$$\frac{(16-16+3)-(3)}{4-0} = 2x-4$$

$$\frac{0}{4} = 2x-4$$

$$4 = 2x$$

$$\boxed{x=2}$$

10. $y = \sqrt{9-8x}$ on $[-2,0]$

$$\frac{\sqrt{9-0} - \sqrt{9+16}}{0 - -2} = \frac{-8}{2\sqrt{9-8x}}$$

$$\frac{3-5}{2} = \frac{-4}{\sqrt{9-8x}}$$

$$-1 = \frac{-4}{\sqrt{9-8x}}$$

$$-\sqrt{9-8x} = -4$$

$$9-8x = 16$$

$$\boxed{x = -7/8}$$

11. **Calculator active problem.** The temperature (in °F) t hours after 9 AM is approximated by the function $T(t) = 50 + 14\sin\frac{\pi t}{12}$. Find the average temperature during the time period 9 AM to 9 PM.

$$\frac{1}{12}\int_0^{12} T(t)\,dt$$

$$\boxed{58.9126\ °F}$$

12. **Calculator active problem.** The depth of water in Mr. Brust's hot tub can be represented by the formula $h(t) = 2 - \cos(t)$, where t is the time in minutes since he begins pouring in water and $h(t)$ is measured in feet. What is the average depth of the water during the first three minutes? Set up the expression and use a calculator to help solve.

$$\frac{1}{3}\int_0^{3} h(t)\,dt$$

$$\boxed{1.953\ ft}$$

13. **Calculator active problem.** The temperature outside during a 12-hour period is given by

$$T(h) = 60 - 5\cos\left(\frac{\pi h}{8}\right), \quad 0 \le h \le 12$$

Where $T(h)$ is measured in degrees Fahrenheit and h is measured in hours. Find the average temperature, to the nearest degree Fahrenheit, between $h = 2$ and $h = 9$.

$$\frac{1}{7}\int_2^9 T(h)\,dh \approx \boxed{61.982\ ^\circ F}$$

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14. Find the number(s) b such that the average value of $y = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal 3. *Hint*: quadratic formula needed!

$$\frac{1}{b-0}\int_0^b (2+6x-3x^2)\,dx = 3$$

$$\frac{1}{b}\left[2x+3x^2-x^3\right]\Big|_0^b = 3$$

$$\frac{1}{b}\left[(2b+3b^2-b^3)-(0)\right] = 3$$

$$2+3b-b^2 = 3$$

$$0 = b^2-3b+1$$

$$b = \frac{3 \pm \sqrt{9-4(1)(1)}}{2(1)}$$

$$\boxed{b = \frac{3 \pm \sqrt{5}}{2}}$$

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15. **Calculator active problem.** Traffic flow is defined as the rate at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled by the function F defined by

$$F(t) = 37 - 6\cos\left(\frac{t}{3}\right) \text{ for } 0 \le t \le 20,$$

where $F(t)$ is measured in cars per minute and t is measured in minutes.

a. What is the average value of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

$$\frac{1}{5}\int_{10}^{15} F(t)\,dt \approx \boxed{39.766 \text{ cars per min.}}$$

b. What is the average rate of change of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

$$\frac{F(15)-F(10)}{15-10} \approx \boxed{-1.518 \text{ cars}/\text{min}^2}$$