6.1 Implicit Differentiation

Calculus

Name: Solutions

Practice

Find $\frac{dy}{dx}$. y' or $\frac{dy}{dx}$

1. $4 = 5x^2 + 2y^3$

$0 = 10x + 6y^2\frac{dy}{dx}$

$-10x = 6y^2\frac{dy}{dx}$

$\boxed{\frac{dy}{dx} = -\frac{5x}{3y^2}}$

2. $5y^2 + 3 = x^2$

$\boxed{\frac{dy}{dx} = \frac{x}{5y}}$

3. $3x = y^3 + 4$

$3 = 3y^2\frac{dy}{dx}$

$\boxed{\frac{dy}{dx} = \frac{1}{y^2}}$

4. $x^2 = 4y^3 + 5y^2$

$\boxed{\frac{dy}{dx} = \frac{x}{6y^2+5y}}$

5. $(4y^3 + 4)^2 = 3x^2$

$2(4y^3+4)\cdot\left(12y^2\frac{dy}{dx}\right) = 6x$

$12y^2\frac{dy}{dx} = \frac{6x}{2(4y^3+4)}$

$\frac{dy}{dx} = \frac{x}{4y^2(4y^3+4)}$

$\boxed{\frac{dy}{dx} = \frac{x}{16y^5+16y^2}}$

6. $2x^3 = (3y^3 + 4)^2$

$\boxed{\frac{dy}{dx} = \frac{x^2}{9y^5+12y^2}}$

7. $-3y + y^3 = 5x$

$-3\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 5$

$\frac{dy}{dx}(-3+3y^2) = 5$

$\boxed{\frac{dy}{dx} = \frac{5}{3y^2-3}}$

8. $5x^3 - 2y = 5y^3$

$\boxed{\frac{dy}{dx} = \frac{15x^2}{2+15y^2}}$

9. $\sin(x + y) = 2x$

$\cos(x+y)\cdot\left(1+\frac{dy}{dx}\right) = 2$

$1+\frac{dy}{dx} = \frac{2}{\cos(x+y)}$

$\boxed{\frac{dy}{dx} = \frac{2}{\cos(x+y)} - 1}$

10. $4x + 1 = \cos y^2$

using y'

$\boxed{y' = -\frac{2}{y}\csc(y^2)}$

11. $3x^2 - 6y^2 + 5 = 9y - 3x$

$6x - 12yy' = 9y' - 3$

$-12yy' - 9y' = -3 - 6x$

$y'(12y+9) = 3+6x$

$\boxed{y' = \frac{6x+3}{12y+9}}$

12. $y^2 - 7y + x^2 - 4x = 10$

$\boxed{y' = \frac{4-2x}{2y-7}}$

13. $e^{y^3} = x^3 + 1$

$e^{y^3} \cdot 3y^2 y' = 3x^2$

$$\boxed{y' = \frac{x^2}{y^2 e^{y^3}}}$$

14. $5x^2 - e^{4y^2} = -6$

$$\boxed{y' = \frac{5x}{4ye^{4y^2}}}$$

15. $\ln(4y^3) = 5x + 3$

$\frac{1}{4y^3} 12y^2 y' = 5$

$\frac{3}{y} y' = 5$

$$\boxed{y' = \frac{5y}{3}}$$

16. $x^3 + 1 = \ln(3y^7)$

$$\boxed{y' = \frac{3x^2 y}{7}}$$

17. $x^3 + y^3 = 6xy$

$3x^2 + 3y^2 \frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$

$3y^2 \frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$

$\frac{dy}{dx}(3y^2 - 6x) = 6y - 3x^2$

$$\boxed{\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}}$$

18. $x^3 - 3x^2y^2 = 3y^3$

$$\boxed{\frac{dy}{dx} = \frac{x^2 - 2xy^2}{3y^2 + 2x^2y}}$$

For 19-23, use implicit differentiation to find $\frac{d^2y}{dx^2}$.

19. $xy = -3$

$y + x\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{y}{x}$

$\frac{d^2y}{dx^2} = -\frac{\frac{dy}{dx}x - y}{x^2}$

$= -\frac{\left(-\frac{y}{x}\right)x - y}{x^2}$

$= -\frac{-2y}{x^2} = \boxed{\frac{2y}{x^2}}$

20. $x^2 + y^2 = 8$

$$\boxed{\frac{d^2y}{dx^2} = -\frac{y^2 + x^2}{y^3}}$$

21. $y^2 = 5x^2 - 3x$

$2y\frac{dy}{dx} = 10x - 3$

$\frac{dy}{dx} = \frac{10x-3}{2y}$

$\frac{d^2y}{dx^2} = \frac{10(2y) - (10x-3)(2\frac{dy}{dx})}{4y^2}$

$= \frac{20y - (10x-3)\left(\frac{10x-3}{y}\right)}{4y^2}$

$\left[20y - \frac{(10x-3)^2}{y}\right] \cdot \frac{1}{4y^2}$

$$\boxed{\frac{20y^2 - (10x-3)^2}{4y^3}}$$

22. $y^3 = x^2 - 4$

$$\boxed{\frac{d^2y}{dx^2} = \frac{6y^3 - 8x^2}{9y^5}}$$

23. $y^2 + 3y = 4x - 5$

$$2yy' + 3y' = 4$$
$$y'(2y+3) = 4$$
$$y' = \frac{4}{2y+3} = 4(2y+3)^{-1}$$
$$y'' = -4(2y+3)^{-2} \cdot 2y'$$
$$y'' = -\frac{4}{(2y+3)^2} \cdot 2 \cdot \frac{4}{2y+3}$$
$$\boxed{y'' = -\frac{32}{(2y+3)^3}}$$

Find the slope of the tangent line at the given point.

24. $2 = 3x^4 + xy^4$ at $(-1, 1)$

$$\boxed{-11/4}$$

25. $x^2 - y^2 = 27$ at $(6, -3)$

$$2x - 2y\frac{dy}{dx} = 0$$
$$2(6) - 2(-3)\frac{dy}{dx} = 0$$
$$12 + 6\frac{dy}{dx} = 0$$
$$\boxed{\frac{dy}{dx} = -2}$$

26. $x \ln y = 4 - 2x$ at $(2, 1)$

$$\boxed{-1}$$

27. $(x - y)^2 - 4x = 20y$ at $(4, 2)$

$$2(x-y)\cdot(1-y') - 4 = 20y'$$
$$2(4-2)(1-y') - 4 = 20y'$$
$$4 - 4y' - 4 = 20y'$$
$$-24y' = 0$$
$$\boxed{y' = 0}$$

Write an equation of the line tangent to the curve at the given point.

28. $x^2 + y^2 + 19 = 2x + 12y$ at $(4, 3)$

$$y - 3 = x - 4$$

or

$$y = x - 1$$

29. $x \sin 2y = y \cos 2x$ at $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$$\sin 2y + x\cos 2y \cdot 2y' = y'\cos 2x + y(-\sin 2x)\cdot 2$$
$$\sin\pi + \tfrac{\pi}{4}\cos\pi \cdot 2y' = y'\cos\tfrac{\pi}{2} - \pi\sin\tfrac{\pi}{2}$$
$$0 + \tfrac{\pi}{4}(-1)\cdot 2y' = y'(0) - \pi(1)$$
$$-\tfrac{\pi}{2}y' = -\pi$$
$$y' = 2 \quad \boxed{y - \tfrac{\pi}{2} = 2\left(x - \tfrac{\pi}{4}\right)}$$

30. Find the points on the curve $x^2 + 2y^2 = 8$ where the tangent line is parallel to the x-axis.

$(0, 2)$ and $(0, -2)$

31. Find the point(s) where the following graph has a **vertical** tangent line. $x + y = y^2$

undefined slope!

$1 + \frac{dy}{dx} = 2y\frac{dy}{dx}$

$\frac{dy}{dx}(1-2y) = -1$

$\frac{dy}{dx} = -\frac{1}{1-2y}$

$1 - 2y = 0$

$y = \frac{1}{2}$

$x + \frac{1}{2} = (\frac{1}{2})^2$

$x = -\frac{1}{4}$

$\boxed{(-\frac{1}{4}, \frac{1}{2})}$

Test Prep: 1E, 2B, 3C, 4C, 5C

Free Response Scoring Guide

Use this only AFTER you have attempted the problem on your own.

Solutions	Points
(a) $g(1) = e^{f(1)} = e^2$ $g'(x) = e^{f(x)}f'(x)$, $g'(1) = e^{f(1)}f'(1) = -4e^2$ The tangent line is given by $y = e^2 - 4e^2(x-1)$.	3 : $\begin{cases} 1: g'(x) \\ 1: g(1) \text{ and } g'(1) \\ 1: \text{tangent line equation} \end{cases}$
(b) $g'(x) = e^{f(x)}f'(x)$ $e^{f(x)} > 0$ for all x So, g' changes from positive to negative only when f' changes from positive to negative. This occurs at $x = -1$ only. Thus, g has a local maximum at $x = -1$.	2 : $\begin{cases} 1: \text{answer} \\ 1: \text{justification} \end{cases}$
(c) $g''(-1) = e^{f(-1)}\left[(f'(-1))^2 + f''(-1)\right]$ $e^{f(-1)} > 0$ and $f'(-1) = 0$ Since f' is decreasing on a neighborhood of -1, $f''(-1) < 0$. Therefore, $g''(-1) < 0$.	2 : $\begin{cases} 1: \text{answer} \\ 1: \text{justification} \end{cases}$
(d) $\frac{g'(3) - g'(1)}{3-1} = \frac{e^{f(3)}f'(3) - e^{f(1)}f'(1)}{2} = 2e^2$	2 : $\begin{cases} 1: \text{difference quotient} \\ 1: \text{answer} \end{cases}$