6.3 Optimization

Calculus

Name: Solutions

Practice

1. What is the smallest product of two numbers given that one number is exactly 7 greater than the other number?

optimize $\longrightarrow$ $P = xy$

$x = y + 7$

$P = (y+7)y$

$P = y^2 + 7y$

$P' = 2y + 7$

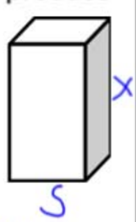
$y = -\frac{7}{2}$ $\quad x = \frac{7}{2}$

$\boxed{P = -\frac{49}{4}}$

2. If the product of two positive numbers is 36, and the sum of the first number plus 4 times the second number is a minimum, what are the two numbers?

$\boxed{3 \text{ and } 12}$

3. A manufacturer wants to design an open box having a square base and a surface area of 108 square inches. What dimensions will produce a box with maximum volume?



$V = lwh$

$l = s$

$w = s$

$h = x$

$V = s^2 x$

$V = s^2\left(\frac{27}{s} - \frac{s}{4}\right)$

$V = 27s - \frac{s^3}{4}$

$0 = V' = 27 - \frac{3}{4}s^2$

$-27 = -\frac{3}{4}s^2$

$36 = s^2$

$6 = s$

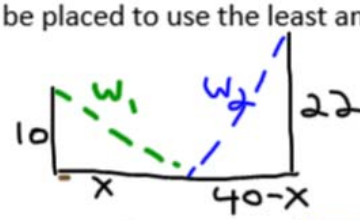
$SA = s^2 + 4sx$

$108 = s^2 + 4sx$

$\frac{27}{s} - \frac{s}{4} = x$

$\boxed{6 \times 6 \times 3 \text{ inches}}$

4. Sullivan and Brust are watching fireworks on the 4th of July. They build towers to get a better view, and decide to work together by securing them to the same stake in the ground. They place the towers 40 feet apart. Mr. Brust's tower is 10 feet high and Mr. Sullivan's tower is 22 feet high. Where should the stake be placed to use the least amount of wire?



$w_1 = \sqrt{x^2 + 100}$ $\qquad w_2 = \sqrt{484 + (40-x)^2}$

Optimize! Use a calculator and set the derivative equal to zero. $\longrightarrow W_1 + W_2$

$\boxed{\text{12.5 feet from Mr. Brust's tower.}}$

5. Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$?

$(x, 4-x^2)$

$d=\sqrt{(x-0)^2+(4-x^2-2)^2}$

$d=\sqrt{x^2+(2-x^2)^2}$

$d=\sqrt{x^4-3x^2+4}$

$d'=\dfrac{4x^3-6x}{2\sqrt{x^4-3x^2+4}}=0$

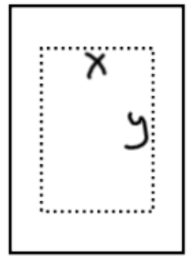
$2x(2x^2-3)=0$

$x=0$ $x=\pm\sqrt{\tfrac{3}{2}}$

max mins

$\boxed{\left(-\sqrt{\tfrac{3}{2}}, \tfrac{5}{2}\right) \quad \left(\sqrt{\tfrac{3}{2}}, \tfrac{5}{2}\right)}$

6. A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be 1.5 inches, and the margins on the left and right are to be 1 inch. What should the dimensions of the page be so that the least amount of paper is used?



$xy=24$

$(x+2)(y+3)=A$

Optimize!

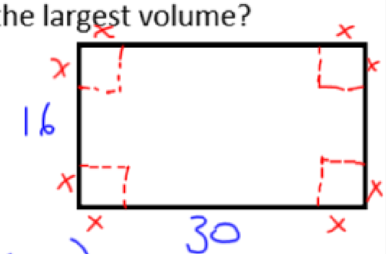
6 x 9 inches

7. You are creating an open-top box with a piece of cardboard that is 16 X 30 inches. What size of square should be cut out of each corner to create a box with the largest volume?

$l=30-2x$

$w=16-2x$

$h=x$



$V=(30-2x)(16-2x)x$

$V=4x^3-92x^2+480x$

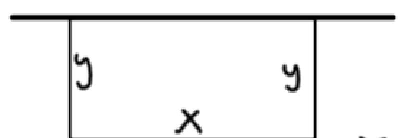
$V'=12x^2-184x+480=0$

$3x^2-46x+120=0$

$x=\dfrac{46\pm\sqrt{2116-4(3)(120)}}{6}=\dfrac{46\pm26}{6}$

$\boxed{x=10/3 \text{ inch}}$

8. Rectangular pig pen using 300 feet of fencing. Built next to an existing wall, so only three sides of fencing needed. What dimensions should the farmer use to construct the pen with the largest possible area?



$x+2y=300$

$xy=A$

Optimize!

75 ft (perpendicular to wall)

x 150 ft (parallel to wall)

9. Which points on the graph of $y = 3 - x^2$ are closest to the point $(0, 1)$? $(x, 3-x^2)$

$d=\sqrt{(x-0)^2+(3-x^2-1)^2}$

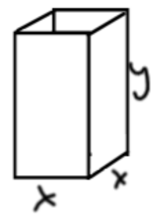
$d=\sqrt{x^4-3x^2+4}$

$d'=\dfrac{4x^3-6x}{2\sqrt{x^4-3x^2+4}}=0$

Same as #5

$\boxed{\left(-\sqrt{\tfrac{3}{2}},\tfrac{3}{2}\right) \quad \left(\sqrt{\tfrac{3}{2}},\tfrac{3}{2}\right)}$

10. Engineers are designing a box-shaped aquarium with a square bottom and an open top. The aquarium must hold 864 cubic feet of water. What dimensions should they use to create an acceptable aquarium with the least amount of glass? → Surface Area (A)



$864 = x^2y$

$A = x^2+4xy$

↑ optimize!

$\boxed{\text{12 x 12 x 6 feet tall}}$

11. What is the radius of a cylindrical soda can with volume of 512 cubic inches that will use the minimum material? Volume of a cylinder is $V = \pi r^2 h$. Surface area of a cylinder is $A = 2\pi r^2 + 2\pi rh$

$512=\pi r^2 h$

$\dfrac{512}{\pi r^2}=h$

$A=2\pi r^2+\left(\dfrac{512}{\pi r^2}\right)(2\pi r)$

$A=2\pi r^2+\dfrac{1024}{r}$

$A'=4\pi r-\dfrac{1024}{r^2}=0$

$4\pi r=\dfrac{1024}{r^2}$

$r^3=\dfrac{256}{\pi}$

$\boxed{r=\sqrt[3]{\dfrac{256}{\pi}}}$

For 12-13, find the largest possible area of each object, given its boundaries. Draw a picture to represent each problem. Give the DIMENSIONS of the object as your final answer.

12. A rectangle is formed in Quadrant I with one corner at the origin and the other corner on the line $y = 8 - 2x$

$A=x\cdot y$

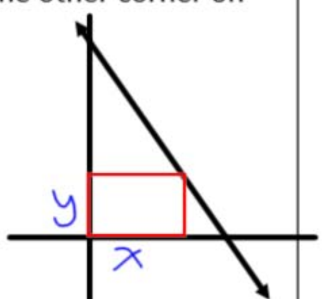
$A=x(8-2x)$

$A=8x-2x^2$

$8-4x=0$

$x=2$

$y=8-2(2)=4$



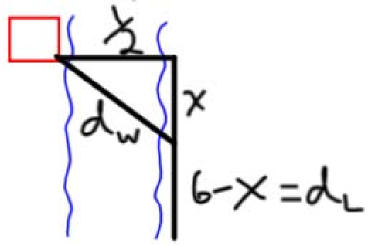
$\boxed{\text{Largest Area is } 2\times 4=8}$

13. A rectangle is formed with the base on the x-axis and the top corners on the function $y = 6 - x^2$

$\boxed{\text{Largest Area is } 2\sqrt{2}\times 4=8\sqrt{2}}$

14. A power station is on one side of a river that is ½ mile wide, and a factory is 6 miles downstream on the other side. It costs $60,000 per mile to run power lines over land and $85,000 per mile to run them underwater. Find the most economical path for the transmission line from the power station to the factory. *Hint*: Total Cost = (on land cost)(distance on land) + (underwater cost)(distance in water)

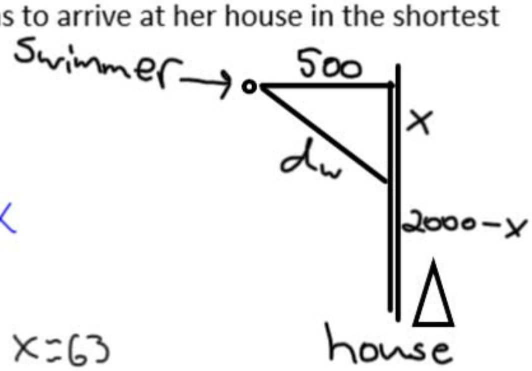
d_L $\qquad$ d_w



After you find your equation to "optimize," take the derivative and use a calculator to help you solve this problem!

$\boxed{x = 0.498 \text{ miles}}$

15. A swimmer is 500 meters from the closest point on a straight shoreline. She needs to reach her house located 2000 meters down shore from the closest point. If she swims at $\frac{1}{2}$ m/s and she runs at 4 m/s, how far from her house should she come ashore so as to arrive at her house in the shortest time? *Hint*: $\text{time} = \frac{\text{distance}}{\text{rate}}$



$$\text{Total Time} = \frac{d_w}{\frac{1}{2}} + \frac{2000-x}{4}$$

$$T = 2\sqrt{500^2+x^2} + 500 - \tfrac{1}{4}x$$

$$T' = \frac{2x}{\sqrt{500^2+x^2}} - \tfrac{1}{4}$$

$$\frac{2x}{\sqrt{500^2+x^2}} = \tfrac{1}{4}$$

Graph! Find pt of intersection

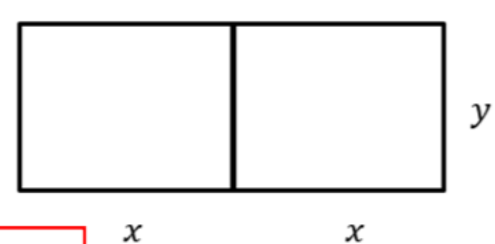
$x \approx 63$

$\boxed{\text{1,937 meters from the house.}}$

16. A rancher has 200 feet of fencing with which to enclose two adjacent rectangular corrals (see figure). What dimensions should be used so that the enclosed area will be a maximum?

$A = 2xy$ $\qquad$ $200 = 4x + 3y$

↑ Optimize!



$x = 25$ $\qquad$ $y = 33\tfrac{1}{3}$ $\qquad$ $\boxed{50 \times 33\tfrac{1}{3} \text{ ft}}$

Test Prep: 1D, 2B, 3A, 4A, 5C