9.3 Average Value of a Function

Calculus

Name: Solutions

Practice

Find the average value of each function on the given interval.

1. $f(x) = x^2$ on $[2, 4]$

$\frac{1}{4-2}\int_2^4 x^2\,dx$

$\frac{1}{2}\cdot\frac{x^3}{3}\Big|_2^4$

$\frac{4^3}{6} - \frac{2^3}{6} = \boxed{\frac{28}{3}}$

2. $f(x) = x^2 - 2x$ on $[0, 3]$

$\boxed{0}$

3. $f(x) = \sin x$ on $[0, \pi]$

$\frac{1}{\pi}\int_0^\pi \sin x\,dx$

$\frac{1}{\pi}(-\cos x)\Big|_0^\pi$

$-\frac{1}{\pi}\cos(\pi) - -\frac{1}{\pi}\cos(0)$

$\frac{1}{\pi} + \frac{1}{\pi} = \boxed{\frac{2}{\pi}}$

4. $f(x) = \sqrt{x}$ on $[0, 16]$

$\boxed{\frac{8}{3}}$

5. $f(x) = \frac{1}{x^2}$ on $[-4, -2]$

$\frac{1}{-2--4}\int_{-4}^{-2} x^{-2}\,dx$

$\frac{1}{2}\cdot\left(-\frac{1}{x}\right)\Big|_{-4}^{-2}$

$\frac{1}{2}\left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{4}\right)$

$\frac{1}{4} - \frac{1}{8} = \boxed{\frac{1}{8}}$

6. $f(x) = 2e^x$ on $[-3, 1]$

$\frac{e}{2} - \frac{1}{2e^3}$

or

$\boxed{\frac{e^4 - 1}{2e^3}}$

On the given interval, find the x-value where the function is equivalent to the average value on that interval.

7. $f(x) = 2x - 2$ on $[1, 4]$

$\frac{1}{3}\int_1^4 (2x-2)\,dx$

$\frac{x^2}{3} - \frac{2x}{3}\Big|_1^4$

$\left(\frac{16}{3} - \frac{8}{3}\right) - \left(\frac{1}{3} - \frac{2}{3}\right)$

$\frac{8}{3} + \frac{1}{3} = \boxed{3}$

$2x - 2 = 3$

$2x = 5$

$\boxed{x = 5/2}$

8. $f(x) = -\frac{x^2}{2}$ on $[0, 3]$

$\boxed{x = \sqrt{3}}$

9. $f(x) = 2x^2 + 16x + 28$ on $[-5, -2]$

$\frac{1}{-2--5}\int_{-5}^{-2} 2x^2 + 16x + 28\,dx$

$\frac{1}{3}\left(\frac{2x^3}{3} + 8x^2 + 28x\right)\Big|_{-5}^{-2}$

$\frac{1}{3}\left(-\frac{16}{3} + 32 - 56\right) - \frac{1}{3}\left(-\frac{250}{3} + 200 - 140\right)$

$-\frac{88}{9} + \frac{70}{9} = \boxed{-2}$

$2x^2 + 16x + 28 = -2$

$x^2 + 8x + 14 = -1$

$(x+5)(x+3) = 0$

$\boxed{x = -5 \text{ and } x = -3}$

Find the average rate of change on the given interval.

10. $f(x) = -(2x - 6)^{\frac{2}{3}}$ on $[1, 3]$

$\boxed{\sqrt[3]{2}}$

11. $y = x^3 - 2x^2 + 2$ on $[-1, 1]$

$\frac{y(1) - y(-1)}{1 - -1} = \frac{1 - -1}{2}$

$\boxed{1}$

12. $y = \ln\sqrt{x}$ on $[1, e]$

$\boxed{\frac{1}{2e - 2}}$

For 13-14, find where the instantaneous rate of change is equivalent to the average rate of change.

13. $y = x^2 - 4x + 3$ on $[0, 4]$

$$\frac{y(4)-y(0)}{4-0} = \frac{3-3}{4} = 0$$

$$y' = 2x - 4$$

$$2x - 4 = 0$$

$$2x = 4$$

$$\boxed{x = 2}$$

14. $y = \sqrt{9 - 3x}$ on $[-2, 3]$

$$\boxed{x = \frac{7}{4}}$$

15. The temperature (in °F) t hours after 9 AM is approximated by the function $T(t) = 50 + 14\sin\frac{\pi t}{12}$. Find the average temperature during the time period 9 AM to 9 PM.

$$\frac{1}{12}\int_0^{12} T(t)\,dt \approx$$

$$58.9126\ °F$$

16. The depth of water in Mr. Brust's hot tub can be represented by the formula $h(t) = -\cos(t) + 2$, where t is the time in minutes since he begins pouring in water and $h(t)$ is measured in feet. What is the average depth of the water during the first three minutes? Set up the expression and use a calculator to help solve.

$$\frac{1}{3}\int_0^{3} h(t)\,dt \approx$$

$$1.9529 \text{ feet}$$

17. Find the number(s) b such that the average value of $y = 2 + 7x - x^3$ on the interval $[0, b]$ is equal 2.

$$\frac{1}{b}\int_0^b (2 + 7x - x^3)\,dx$$

$$\frac{1}{b}\left(2x + \frac{7}{2}x^2 - \frac{x^4}{4}\right)\Big|_0^b = 2$$

$$\frac{1}{b}\left(2b + \frac{7}{2}b^2 - \frac{b^4}{4}\right) - \frac{1}{b}(0) = 2$$

$$2 + \frac{7}{2}b - \frac{b^3}{4} = 2$$

$$b\left(\frac{7}{2} - \frac{b^2}{4}\right) = 0$$

$$\cancel{b = 0} \quad \text{or} \quad \boxed{b = \pm\sqrt{14}}$$

18. Find the number(s) b such that the average value of $y = 2 + 6x - 3x^2$ on the interval $[0, b]$ is equal 3. *Hint*: quadratic formula needed!

$$\boxed{b = \frac{3 \pm \sqrt{5}}{2}}$$

19. 2004 A Q1 c-d

Traffic flow is defined as the rate at which cars pass through an intersection, measured in cars per minute. The traffic flow at a particular intersection is modeled by the function F defined by

$$F(t) = 82 + 4\sin\left(\frac{t}{2}\right) \text{ for } 0 \le t \le 30,$$

where $F(t)$ is measured in cars per minute and t is measured in minutes.

(c) What is the average value of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

$$\boxed{\frac{1}{5}\int_{10}^{15} F(t)\,dt = 81.899 \text{ cars/min}}$$

(d) What is the average rate of change of the traffic flow over the time interval $10 \le t \le 15$? Indicate units of measure.

$$\boxed{\frac{F(15) - F(10)}{15 - 10} = 1.517 \text{ or } 1.518 \text{ cars/min}^2}$$

20. 2005 A Q3 b-d

Distance x (cm)	0	1	5	6	8
Temperature $T(x)$ (°C)	100	93	70	62	55

A metal wire of length 8 centimeters (cm) is heated at one end. The table above gives selected values of the temperature $T(x)$, in degrees Celsius (°C), of the wire x cm from the heated end. The function T is decreasing and twice differentiable.

(b) Write an integral expression in terms of $T(x)$ for the average temperature of the wire. Estimate the average temperature of the wire using a trapezoidal sum with the four subintervals indicated by the data in the table. Indicate units of measure.

$\frac{1}{8}\int_0^8 T(x)\,dx$

Trapezoidal approximation for $\int_0^8 T(x)\,dx$:

$$A = \frac{100+93}{2}\cdot 1 + \frac{93+70}{2}\cdot 4 + \frac{70+62}{2}\cdot 1 + \frac{62+55}{2}\cdot 2$$

Average temperature $\approx \frac{1}{8}A = 75.6875$°C

(c) Find $\int_0^8 T'(x)\,dx$, and indicate units of measure. Explain the meaning of $\int_0^8 T'(x)\,dx$ in terms of the temperature of the wire.

$\int_0^8 T'(x)\,dx = T(8) - T(0) = 55 - 100 = -45$°C

The temperature drops 45°C from the heated end of the wire to the other end of the wire.

(d) Are the data in the table consistent with the assertion that $T''(x) > 0$ for every x in the interval $0 < x < 8$? Explain your answer.

Average rate of change of temperature on $[1, 5]$ is $\frac{70-93}{5-1} = -5.75$.

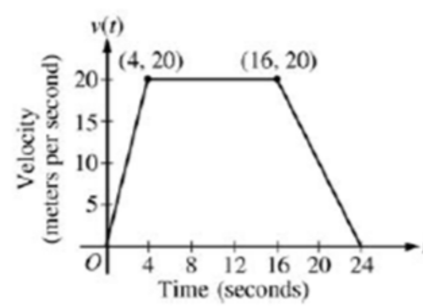
Average rate of change of temperature on $[5, 6]$ is $\frac{62-70}{6-5} = -8$.

No. By the MVT, $T'(c_1) = -5.75$ for some c_1 in the interval $(1, 5)$ and $T'(c_2) = -8$ for some c_2 in the interval $(5, 6)$. It follows that T' must decrease somewhere in the interval (c_1, c_2). Therefore T'' is not positive for every x in $[0, 8]$.

---

21. 2005 A Q5 d

A car is traveling on a straight road. For $0 \le t \le 24$ seconds, the car's velocity $v(t)$, in meters per second, is modeled by the piecewise-linear function defined by the graph to the right.



(d) Find the average rate of change of v over the interval $8 \le t \le 20$. Does the Mean Value Theorem guarantee a value of c, for $8 < c < 20$, such that $v'(c)$ is equal to this average rate of change? Why or why not?

The average rate of change of v on $[8, 20]$ is $\frac{v(20)-v(8)}{20-8} = -\frac{5}{6}$ m/sec^2.

No, the Mean Value Theorem does not apply to v on $[8, 20]$ because v is not differentiable at $t = 16$.

Test Prep: 1E, 2B, 3A, 4C, 5C