1.8 The Squeeze Theorem

Calculus

Solutions **Practice**

Evaluate each limit.

1. $\lim_{x\to 0} x\cos\left(\frac{1}{x}\right)$

$-1 \le \cos\left(\frac{1}{x}\right) \le 1$

$-x \le x\cos\left(\frac{1}{x}\right) \le x$

$0 \le \lim_{x\to 0} x\cos\left(\frac{1}{x}\right) \le 0$

Therefore

$\lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0$

2. $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x^2}\right)$

$-1 \le \sin\left(\frac{1}{x^2}\right) \le 1$

$-x^2 \le x^2\sin\left(\frac{1}{x^2}\right) \le x^2$

$0 \le \lim_{x\to 0} x^2\sin\left(\frac{1}{x^2}\right) \le 0$

Therefore

$\lim_{x\to 0} x^2\sin\left(\frac{1}{x^2}\right) = 0$

3. $\lim_{x\to 0} x\sin\left(\frac{1}{x^2}\right)$

$-1 \le \sin\left(\frac{1}{x^2}\right) \le 1$

$-x \le x\sin\left(\frac{1}{x^2}\right) \le x$

$0 \le \lim_{x\to 0} x\sin\left(\frac{1}{x^2}\right) \le 0$

Therefore

$\lim_{x\to 0} x\sin\left(\frac{1}{x^2}\right) = 0$

4. Let g and h be the functions defined by $g(x) = x^2 - 3x$ and $h(x) = 2 - 2x$. If f is a function that satisfies $g(x) \le f(x) \le h(x)$ for all x, what is $\lim_{x\to 2} f(x)$?

$2^2 - 3(2) \le \lim_{x\to 2} f(x) \le 2 - 2(2)$

$-2 \le \lim_{x\to 2} f(x) \le -2$

$\boxed{\lim_{x\to 2} f(x) = -2}$

5. Let g and h be the functions defined by $g(x) = \cos(\pi(x+2)) - 3$ and $h(x) = \frac{x^2}{2} + x - \frac{7}{2}$. If f is a function that satisfies $g(x) \le f(x) \le h(x)$ for $-2 \le x \le 0$, what is $\lim_{x\to -1} f(x)$?

$\cos(\pi(-1+2)) - 3 \le \lim_{x\to -1} f(x) \le \frac{(-1)^2}{2} + (-1) - \frac{7}{2}$

$\cos(\pi) - 3 \le \lim_{x\to -1} f(x) \le -\frac{6}{2} - 1$

$-4 \le \lim_{x\to -1} f(x) \le -4$

$\boxed{\lim_{x\to -1} f(x) = -4}$

6. Let g and h be the functions defined by $g(x) = x^2 + x - 1$ and $h(x) = -x^2 - 4x - 2$. If f is a function that satisfies $g(x) \le f(x) \le h(x)$ for all x, what is $\lim_{x\to -1} f(x)$?

$(-1)^2 + (-1) - 1 \le \lim_{x\to -1} f(x) \le -(-1)^2 - 4(-1) - 2$

$1 - 2 \le \lim_{x\to -1} f(x) \le -1 + 4 - 2$

$-1 \le \lim_{x\to -1} f(x) \le 1$

Limit cannot be determined from the Squeeze Theorem.

7. Let g and h be the functions defined by $g(x) = -x^2 - 2x + 5$ and $h(x) = 2x^2 - x - 4$. If f is a function that satisfies $g(x) \le f(x) \le h(x)$ for all x, what is $\lim_{x\to -2} f(x)$?

$-(-2)^2 - 2(-2) + 5 \le \lim_{x\to -2} f(x) \le 2(-2)^2 - (-2) - 4$

$-4 + 4 + 5 \le \lim_{x\to -2} f(x) \le 8 + 2 - 4$

$5 \le \lim_{x\to -2} f(x) \le 6$

Limit cannot be determined from the Squeeze Theorem.

8. Let g and h be the functions defined by $g(x) = \sin\left(\frac{\pi}{2}(x+1)\right) - 1$ and $h(x) = \cos(\pi x) - 3$. If f is a function that satisfies $g(x) \le f(x) \le h(x)$ for all x, what is $\lim_{x\to 2} f(x)$?

$\sin\left(\frac{\pi}{2}(3)\right) - 1 \le \lim_{x\to 2} f(x) \le \cos(2\pi) - 3$

$-1 - 1 \le \lim_{x\to 2} f(x) \le 1 - 3$

$-2 \le \lim_{x\to 2} f(x) \le -2$

$\boxed{\lim_{x\to 2} f(x) = -2}$

9. Let g and h be the functions defined by $g(x) = x^2$ and $h(x) = \cos(x)$. If f is a function that satisfies $g(x) \le f(x) \le h(x)$ for $-0.4 \le x \le 0.4$, what is $\lim_{x\to 0} f(x)$?

$0^2 \le \lim_{x\to 0} f(x) \le \cos(0)$

$0 \le \lim_{x\to 0} f(x) \le 1$

Limit cannot be determined from the Squeeze Theorem.

10. Let f and g be the functions defined by $f(x) = \frac{\sin x}{5x}$ and $g(x) = x^2 \cos\left(\frac{1}{x^3}\right)$ for $x \neq 0$. The following inequalities are true for $x \neq 0$. State whether each inequality can be used with the squeeze theorem to find the limit of the function as x approaches 0?

a. $5(\sin(\pi(x+1))) \leq f(x) \leq \frac{1}{5}$

$5(\sin(\pi)) \leq \lim_{x \to 0} f(x) \leq \frac{1}{5}$

$0 \leq \lim_{x \to 0} f(x) \leq \frac{1}{5}$

No

b. $-x^2 \leq g(x) \leq x^2$

$0 \leq \lim_{x \to 0} g(x) \leq 0$

Yes

11. Let f and g be the functions defined by $f(x) = \frac{\cos x - 1}{x^2}$ and $g(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$. The following inequalities are true for $x \neq 0$. State whether each inequality can be used with the squeeze theorem to find the limit of the function as x approaches 0?

a. $\frac{1}{5} \leq f(x) \leq \frac{1}{2}$

No

b. $\frac{1}{2} - x^2 \leq f(x) \leq \frac{1}{2} + x^2$

$\frac{1}{2} \leq \lim_{x \to 0} f(x) \leq \frac{1}{2}$

Yes

c. $-x^2 \leq g(x) \leq x^2$

$0 \leq \lim_{x \to 0} g(x) \leq 0$

Yes

d. $-\frac{1}{x} \leq g(x) \leq \frac{1}{x}$

No. $\frac{1}{x}$ is unbounded as $x \to 0$.

12. Let f and g be the functions defined by $f(x) = \frac{x^2 \sin x}{x}$ and $g(x) = x \cos\left(\frac{1}{|x|}\right)$ for $x \neq 0$. The following inequalities are true for $x \neq 0$. State whether each inequality can be used with the squeeze theorem to find the limit of the function as x approaches 0?

a. $-x \leq f(x) \leq x$

$0 \leq \lim_{x \to 0} f(x) \leq 0$

Yes

b. $-\frac{1}{x} \leq g(x) \leq \frac{1}{x}$

No, $\frac{1}{x}$ is unbounded as $x \to 0$.

c. $2^{-x} \leq g(x) \leq 2^x$

$2^0 \leq \lim_{x \to 0} g(x) \leq 2^0$

$1 \leq \lim_{x \to 0} g(x) \leq 1$

Yes