10.10 Alternating Series Error Bound

Calculus

Solutions

Practice

A calculator may be used on all problems in this practice. For 1-2, approximate an interval of the sum of the alternating series using the Alternating Series Error Bound for the first 6 terms.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3^n}$

$\displaystyle\sum_{n=1}^{6} a_n \approx 0.185185$

$|a_7| = \frac{7}{3^7} \approx 0.0032$

$|S - S_6| \le 0.032$

$-0.0032 \le S - 0.185185 \le 0.0032$

$\boxed{0.1819 \le S \le 0.188}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4}{\ln(n+2)}$

$\displaystyle\sum_{n=1}^{6} a_n \approx 1.14046$

$|a_7| = \frac{4}{\ln(9)} \approx 1.8204$

$|S - S_6| \le 1.8204$

$-1.8204 \le S - 1.14046 \le 1.8204$

$\boxed{-0.68 \le S \le 2.9609}$

3. Determine the number of terms needed to approximate the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ with an error less than 10^{-3}.

$|a_{n+1}| < 10^{-3}$

$\frac{1}{(n+1)^2} < \frac{1}{1000}$

$1000 < (n+1)^2$

$\sqrt{1000} < n+1$

$n > 30.6227$

$\boxed{31 \text{ terms}}$

4. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ converges to S. Using the alternating series bound, what is the least number of terms that must be summed to guarantee a partial sum that is within 0.05 of S?

$|a_{n+1}| < 0.05$

This is tricky! "Within" would mean not equal to 0.05, but less than 0.05

$\frac{1}{\sqrt{n+1}} < 0.05$

$20 < \sqrt{n+1}$

$20^2 < n+1$

$399 < n$

(A) 20

(B) 55

(C) 399

(D) 400

5. If the infinite series $S = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{4}{n}$ is approximated by $P_k = \sum_{n=1}^{k}(-1)^{n+1}\frac{4}{n}$, what is the least value of k for which the alternating series error bound guarantees that $|S - P_k| < \frac{7}{100}$?

$|S-P_k| \le |a_{k+1}| < \frac{7}{100}$

$\frac{4}{k+1} < \frac{7}{100}$

$400 < 7k+7$

$\frac{393}{7} < k$

$\to k > 56.1428$

(A) 55 (B) 56 (C) 57 (D) 60

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6. If the series $S = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^3}$ is approximated by the partial sum $S_k = \sum_{n=1}^{k}(-1)^{n+1}\frac{1}{n^3}$, what is the least value of k for which the alternating series error bound guarantees that $|S - S_k| \le \frac{7}{10000}$?

$|S-S_k| \le |a_{k+1}| \le \frac{7}{10{,}000}$

$\frac{1}{(k+1)^3} \le \frac{7}{10{,}000}$

$\frac{10{,}000}{7} \le (k+1)^3$

$\sqrt[3]{\frac{10{,}000}{7}} \le k+1$

$\to k \ge 10.262$

(A) 10 (B) 11 (C) 12 (D) 13

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7. The series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ converges by the alternating series test. If $S_n = \sum_{k=1}^{n}(-1)^{k+1}a_k$ is the nth partial sum of the series, which of the following statements must be true?

Might be $> a_{26}$

(A) $\lim_{n\to\infty} S_n = 0$ (B) $\lim_{n\to\infty} a_n = S$ (C) $|S - S_{20}| \le a_{26}$ (D) $|S - S_{25}| \le a_{26}$

8. If the series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{5n+1}$ is approximated by the partials sum with 15 terms, what is the alternating series error bound?

$$|a_{n+1}| = \frac{1}{5(n+1)+1} = \frac{1}{5(16)+1} = \frac{1}{80+1}$$

(A) $\frac{1}{15}$ (B) $\frac{1}{16}$ (C) $\frac{1}{76}$ (D) $\frac{1}{81}$

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9. The function f is defined by the power series $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x}{(2n+1)!}$ for all real numbers x. Show that $1 - \frac{1}{3!} + \frac{1}{5!}$ approximates $f(1)$ with an error less than $\frac{1}{4000}$.

$$|S - S_n| < |a_{n+1}|$$

Next term = error bound

$$\left| f(1) - \left[1 - \frac{1}{3!} + \frac{1}{5!}\right] \right| < \frac{1}{7!}$$

$\frac{1}{7!} = \frac{1}{5040}$, which is $< \frac{1}{4000}$

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Test Prep

10. **Calculator active!** Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n n^n}{n!}$ for all x for which the series converges.

a. Use the first three terms of the series to approximate $f\left(-\frac{1}{3}\right)$.

$$f\left(-\tfrac{1}{3}\right) \approx \sum_{n=1}^{3} \frac{\left(-\frac{1}{3}\right)^n \cdot n^n}{n!} = -0.2778$$

b. How far off is this estimate from the value of $f\left(-\frac{1}{3}\right)$? Justify your answer.

Error bound is the next term.

$$a_4 = \frac{\left(-\frac{1}{3}\right)^4 \cdot 4^4}{4!} \approx 0.13168$$

The estimate of -0.2778 is off by 0.13168 or less. The error bound is the next term a_4.

11. If the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ is approximated with the series $\underbrace{\sum_{n=1}^{7}(-1)^{n+1}\frac{1}{n^2}}_{7^{th}\text{ term}}$, what is the error bound?

Error bound = 8^{th} term

$$|a_8| = \frac{1}{8^2} = \boxed{\frac{1}{64}}$$