1. Find the fourth-degree Maclaurin Polynomial for e^{4x}.

$c=0$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!}$$

$f(x) = e^{4x}$ $\quad f(0) = 1$

$f'(x) = 4e^{4x}$ $\quad f'(0) = 4$

$f''(x) = 16e^{4x}$ $\quad f''(0) = 16$

$f'''(x) = 64e^{4x}$ $\quad f'''(0) = 64$

$f^{(4)}(x) = 256e^{4x}$ $\quad f^{(4)}(0) = 256$

$$P_4(x) = 1 + 4x + \frac{16x^2}{2} + \frac{64x^3}{3!} + \frac{256x^4}{4!}$$

Simplified

$$\boxed{P_4(x) = 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4}$$

2. Find the fifth-degree Maclaurin Polynomial for the function $f(x) = \sin x$.

$c=0$

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$f = \sin x$ $\quad f(0) = 0$

$f' = \cos x$ $\quad f'(0) = 1$

$f'' = -\sin x$ $\quad f''(0) = 0$

$f''' = -\cos x$ $\quad f'''(0) = -1$

$f^{(4)} = \sin x$ $\quad f^{(4)}(0) = 0$

$f^{(5)} = \cos x$ $\quad f^{(5)}(0) = 1$

$$P_5(x) = 0 + 1\cdot x + 0x^2 + \frac{-1}{3!}x^3 + 0x^4 + \frac{1}{5!}x^5$$

$$\boxed{P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5}$$

3. Find the third-degree Taylor Polynomial for $f(x) = \ln(2x)$ about $x = 1$.

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2} + \frac{f'''(1)(x-1)^3}{3!}$$

$f(x) = \ln(2x)$ $\quad f(1) = \ln 2$

$f'(x) = \frac{1}{x}$ $\quad f'(1) = 1$

$f''(x) = -\frac{1}{x^2}$ $\quad f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$ $\quad f'''(1) = 2$

$$P_3(x) = \ln 2 + (x-1) + -\frac{1}{2}(x-1)^2 + \frac{2}{3!}(x-1)^3$$

Simplified

$$\boxed{P_3(x) = \ln 2 + x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3}$$

4. Find the third-degree Taylor Polynomial about $x = 0$ for $\ln(1 - x)$.

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)(x-0)^2}{2} + \frac{f'''(0)(x-0)^3}{3!}$$

$f(x) = \ln(1-x)$ $\quad f(0) = 0$

$f'(x) = -\frac{1}{1-x}$ $\quad f'(0) = -1$

$f''(x) = -\frac{1}{(1-x)^2}$ $\quad f''(0) = -1$

$f'''(x) = -\frac{2}{(1-x)^3}$ $\quad f'''(0) = -2$

$$P_3(x) = 0 + (-1)x + \frac{(-1)x^2}{2} + \frac{(-2)x^3}{6}$$

$$\boxed{P_3(x) = -x - \frac{x^2}{2} - \frac{x^3}{3}}$$

5. Find the third-degree Taylor Polynomial about $x = 4$ of $\sqrt{x}$.

$$P_3(x) = f(4) + f'(4)(x-4) + \frac{f''(4)(x-4)^2}{2} + \frac{f'''(4)(x-4)^3}{3!}$$

$f(x) = \sqrt{x}$ $\quad f(4) = 2$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\quad f'(4) = \frac{1}{4}$

$f''(x) = -\frac{1}{4x^{3/2}}$ $\quad f''(4) = -\frac{1}{32}$

$f'''(x) = \frac{3}{8x^{5/2}}$ $\quad f'''(4) = \frac{3}{256}$

$$P_3(x) = 2 + \frac{1}{4}(x-4) + \left(-\frac{1}{32}\right)\left(\frac{1}{2}\right)(x-4)^2 + \left(\frac{3}{256}\right)\left(\frac{1}{6}\right)(x-4)^3$$

$$\boxed{P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3}$$

6. The function f has derivatives of all orders for all real numbers with $f(1) = -1$, $f'(1) = 4$, $f''(1) = 6$, and $f'''(1) = 12$. Using the third-degree polynomial for f about $x = 1$, what is the approximation of $f(1.1)$?

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2} + \frac{f'''(1)(x-1)^3}{3!}$$

$$P_3(x) = -1 + 4(x-1) + \frac{6(x-1)^2}{2} + \frac{12(x-1)^3}{6}$$

$$f(1.1) \approx P_3(1.1) = \boxed{-0.568}$$

$f(0) \quad f'(0)$

7. A function f has a Maclaurin series given by $3 + 4x + 2x^2 + \frac{1}{3}x^3 + \cdots$, and the series converges to $f(x)$ for all real numbers x. If g is the function defined by $g(x) = e^{f(x)}$, what is the coefficient of x in the Maclaurin series for g?

$$P_n(x) = g(0) + \underline{g'(0)}\,x + g''(0)\frac{x^2}{2} + \cdots$$

$$g'(x) = f'(x)e^{f(x)}$$

$$g'(0) = f'(0)e^{f(0)}$$

$$\boxed{4e^3}$$

8. Let f be a function with third derivative $f'''(x) = (7x+2)^{\frac{3}{2}}$. What is the coefficient of $(x-2)^4$ in the fourth-degree Taylor Polynomial for f about $x = 2$?

$$f^4(x) = \frac{3}{2} \cdot 7(7x+2)^{1/2}$$

$$f^4(2) = \frac{21}{2}\sqrt{16} = 42$$

$$\frac{f''''(2)(x-2)^4}{4!}$$

$$\frac{42}{24} = \boxed{\frac{7}{4}}$$

9. Let $P(x) = 4x^2 - 6x^3 + 8x^4 + 4x^5$ be the fifth-degree Taylor Polynomial for the function f about $x = 0$. What is the value of $f'''(0)$?

$$\frac{f'''(0)}{3!} = -6$$

$$\boxed{f'''(0) = -36}$$

10. Let P be the second-degree Taylor Polynomial for $f(x) = e^{-3x}$ about $x = 3$. What is the slope of the line tangent to the graph of P at $x = 3$?

At $x = 3$, the slope of P and the slope of f are the same!

$$P'(3) = f'(3)$$

$$f'(x) = -3e^{-3x}$$

$$f'(3) = \boxed{-3e^{-9}}$$

11. Let f be a function with $f(4) = 2$, $f'(4) = -1$, $f''(4) = 6$, and $f'''(4) = 12$. What is the third-degree Taylor Polynomial for f about $x = 4$?

$$P_3(x) = f(4) + f'(4)(x-4) + \frac{f''(4)(x-4)^2}{2} + \frac{f'''(4)(x-4)^3}{3!}$$

$$P_3(x) = 2 + (-1)(x-4) + \frac{6}{2}(x-4)^2 + \frac{12}{3!}(x-4)^3$$

$$\boxed{P_3(x) = 2 - (x-4) + 3(x-4)^2 + 2(x-4)^3}$$

12. Let f be a function that has derivates of all orders for all real numbers. Assume $f(1) = 3$, $f'(1) = -2$, $f''(1) = 2$, and $f'''(1) = 4$. Use a second-degree Taylor Polynomial to approximate $f(0.7)$.

$$P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2}$$

$$P_2(x) = 3 - 2(x-1) + (x-1)^2$$

$$f(0.7) \approx P_2(0.7) = \boxed{3.690}$$

13. The function f has derivatives of all orders for all real numbers with $f(0) = 4$, $f'(0) = -3$, $f''(0) = 3$, and $f'''(0) = 2$. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. Find the third-degree Taylor Polynomial for g about $x = 0$.

$$P_3(x) = g(0) + g'(0)x + \frac{g''(0)x^2}{2} + \frac{g'''(0)x^3}{3!}$$

$g(x) = \int_0^x f(t)\,dt$ $\quad g(0) = 0$

$g'(x) = f(x)$ $\quad g'(0) = 4$

$g''(x) = f'(x)$ $\quad g''(0) = -3$

$g'''(x) = f''(x)$ $\quad g'''(0) = 3$

$$P_3(x) = 0 + 4x + \frac{-3}{2}x^2 + \frac{3}{3!}x^3$$

$$\boxed{P_3(x) = 4x - \frac{3}{2}x^2 + \frac{1}{2}x^3}$$

14.

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
-4	1	-2	-4	2

Selected values for $f(x)$ and its first three derivatives are shown in the table above. What is the approximation for the value of $f(-3)$ about $x = -4$ obtained using the third-degree Taylor Polynomial for f.

$$P_3(x) = f(-4) + f'(-4)(x+4) + \frac{f''(-4)}{2}(x+4)^2 + \frac{f'''(-4)}{3!}(x+4)^3$$

$$P_3(x) = 1 - 2(x+4) + \frac{-4}{2}(x+4)^2 + \frac{2}{6}(x+4)^3$$

$$f(-3) \approx P_3(-3) = \boxed{-\frac{8}{3}}$$

10.11 Taylor Polynomial Approximations

Test Prep

15. Which of the following polynomial approximations is the best for $\cos(3x)$ near $x = 0$?

$f(x) = \cos(3x)$ $\quad f(0) = 1$

$f'(x) = -3\sin(3x)$ $\quad f'(0) = 0$

$f''(x) = -9\cos(3x)$ $\quad f''(0) = -9$

$$P(x) = 1 + 0x + \frac{-9}{2}x^2$$

(A) $1 + \frac{3}{2}x$ (B) $1 - \frac{9}{2}x^2$ (C) $1 + x$ (D) $1 - \frac{9}{2}x + x^2$

16. Consider the logistic differential equation $\frac{dy}{dt} = \frac{y}{6}(4 - y)$. Let $y = f(t)$ be the particular solution to the differential equation with $f(0) = 6$.

a. Write the second-degree Taylor Polynomial for f about $t = 0$.

$f(0) = 6$

$f'(0) = \left.\frac{dy}{dt}\right|_{t=0} = \frac{6}{6}(4-6) = -2$

$f''(0) = \left.\frac{d^2y}{dt^2}\right|_{t=0} = \left.\frac{4}{6}\frac{dy}{dt} - \frac{2}{6}y\frac{dy}{dt}\right|_{t=0}$

$= \frac{2}{3}(-2) - \frac{1}{3}(6)(-2)$

$= -\frac{4}{3} + 4 = \frac{8}{3}$

$P_2(t) = f(0) + f'(0)\,t + \frac{f''(0)}{2}t^2$

$P_2(t) = 6 + (-2) + \frac{8/3}{2}\ 2$

$\boxed{P_2(t) = 6 - 2t + \frac{4}{3}t^2}$

b. Use the results from part a to approximate $f(1)$.

$f(1) \approx P_2(1) = 6 - 2 + \frac{4}{3} = \boxed{\frac{16}{3}}$

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t (seconds)	0	4	10
$x'(t)$ meters per second	5.0	5.8	4.0

17. The position of a particle moving along a straight line is modeled by $x(t)$. Selected values of $x'(t)$ are shown in the table above and the position of the particle at time $t = 10$ is $x(10) = 8$.

a. Approximate $x''(8)$ using the average rate of change of $x'(t)$ over the interval $4 \le t \le 10$. Show computations that lead to your answer.

$\frac{x'(10) - x'(4)}{10 - 4} = \frac{4.0 - 5.8}{10 - 4} = \frac{-1.8}{6} = \boxed{-0.3}$

b. Using correct units, explain the meaning of $x''(8)$ in the context of the problem.

$x''(8)$ is the rate of change of the rate of change of the position of the particle, in meters per second per second, at time $t = 8$ seconds.

c. Use a right Riemann sum with two subintervals to approximate $\int_0^{10} |x'(t)|\, dt$.

$(4)(5.8) + (6)(4.0) = \boxed{47.2}$

Both positive. Abs. value does not change anything.

d. Let s be a function such that the third derivative of s with respect to t is $(t - 3)^7$. Write the fourth-degree term of the fourth-degree Taylor Polynomial for s about $t = 1$.

$s^{(4)}(x) = 7(t-3)^6$

$s^{(4)}(1) = 7(-2)^6 = 448$

$\frac{448(t-1)^4}{4!} = \boxed{\frac{56}{3}(t-1)^4}$