Calculus

10.12 Lagrange Error Bound

Notes

Write your questions and thoughts here!

Exact value = Approximate value + Remainder

Error:

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \cdots + \frac{f^{(n)}(c)(x-c)^n}{n!} + R(x)$$

Lagrange Error Bound

Let $f(x)$ be differentiable through the order $n+1$. The error between the Taylor Polynomial and $f(x)$ is bounded by:

$$|R_n(x)| \leq$$

where z is some number between c and x.

1. The fourth degree Maclaurin polynomial for $\cos x$ is given by $p_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$. If this polynomial is used to approximate $\cos(0.2)$, what is the Lagrange error bound?

Write your questions and thoughts here!

2. Use a third degree Taylor polynomial on the interval $[0, 1]$ for e^x centered about $x = 0$ to approximate e^1. What is the error bound of this approximation?

3. What is the smallest order Taylor Polynomial centered at $x = 1$ which will approximate e^{x-1} on the interval $[0, 3]$ with a Lagrange error bound less than 1?

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Calculus

Practice

1. The third Maclaurin polynomial for $\sin x$ is given by $p(x) = x - \frac{x^3}{3!}$. If this polynomial is used to approximate $\sin(0.1)$, what is the Lagrange error bound?

2. If the Taylor Polynomial for approximating $\cos x$ is given by $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, what is the upper bound for the error in the approximation of $\cos(0.3)$?

3. If the Taylor Polynomial about $x = 0$ for the approximation of e^x is given by $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$, what is the upper bound for the error in the approximation of e?

4. Let f be a function that has derivatives of all orders for all real numbers and let $P_3(x)$ be the third-degree Taylor Polynomial for f about $x = 0$. $\left|f^{(n)}(x)\right| \leq \frac{n}{n+1}$, for $1 \leq n \leq 5$ and all values of x. Of the following, which is the smallest value of k for which the Lagrange error bound guarantees that $|f(1) - P_3(1)| \leq k$?

(A) $\frac{5}{6}$ (B) $\frac{5}{6} * \frac{1}{5!}$ (C) $\frac{5}{6} * \frac{1}{4!}$ (D) $\frac{4}{5} * \frac{1}{4!}$

5. The function f has derivatives of all orders for all real numbers, $f^{(4)}(x) = e^{\cos x}$. If the third-degree Taylor Polynomial for f about $x = 0$ is used to approximate f on the interval [0,1], what is the Lagrange error bound?

6. The Taylor series for a function f about $x = 3$ is given by $\sum_{n=0}^{\infty}(-1)^n \frac{3n+1}{2^n}(x-3)^n$ and converges to f for $0 \le x \le 5$. If the third-degree Taylor Polynomial for f about $x = 3$ is used to approximate $f\left(\frac{13}{4}\right)$, what is the alternating series error bound?

7. Let f be a polynomial function with nonzero coefficients such that $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$. $T_4(x)$ is the fourth-degree Taylor Polynomial for f about $x = c$ such that $T_4 = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + b_4(x-c)^4$. Based on the Lagrange error bound, $f(x) - T_4(x)$ must equal which of the following?

(A) x (B) $(x-c)^5$ (C) $a_5(x-c)^5$ (D) $\frac{a_5(x-c)^5}{5!}$

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8. Let $P(x)$ be the sixth-degree Taylor Polynomial for a function f about $x = 0$. Information about the maximum of the absolute value of selected derivatives of f over the interval $0 \le x \le 1.5$ is given below.

$\max_{0\le x\le 1.5}\left|f^{(5)}(x)\right| = 9.3$ $\max_{0\le x\le 1.5}\left|f^{(6)}(x)\right| = 62.1$ $\max_{0\le x\le 1.5}\left|f^{(7)}(x)\right| = 481.3$

What is the smallest value of k for which the Lagrange error bound guarantees that $|f(1.5) - P(1.5)| \le k$?

9. The function f has derivatives of all orders for all real numbers. Values of f and its first four derivatives at $x = 2$ are given in the table.

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$
2	6	-12	18	-24	34

a. Write the third-degree Taylor Polynomial for f about $x = 2$, and use it to approximate $f(1.5)$.

b. The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \le 48$ for all $x > 1$. Use the Lagrange error bound to show that the approximation found in part (a) differs from $f(1.5)$ by no more than $\frac{1}{8}$.

10. Let h be a function having derivatives of all orders for $x > 0$. Selected values for the first four derivatives of h are given for $x = 3$. Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 3$ approximates $h(2.9)$ with an error less than 3×10^{-4}.

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
3	317	$\frac{753}{4}$	$\frac{1383}{4}$	$\frac{3483}{8}$	$\frac{1125}{16}$

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Test Prep

11. **Calculator allowed.**

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$
3	4	-8	14	-22	30

The function f has derivatives of all orders for all real numbers. Values of f and its first four derivatives at $x = 3$ are given in the table.

a. Write an equation for the line tangent to the graph of f at $x = 3$ and use it to approximate $f(2.5)$.

b. Write the third-degree Taylor polynomial for f about $x = 3$, and use it to approximate $f(2.5)$.

c. Is there enough information to determine whether f has a critical point at $x = 2.5$? If not, explain why not. If so, determine whether $f(2.5)$ is a relative maximum, relative minimum, or neither, and give a reason for your answer.

d. The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \leq 48$ for all $x > 2$. Use the Lagrange error bound to show that the approximation found in part (b) differs from $f(2.5)$ by no more than $\frac{1}{8}$.

e. What is the coefficient of the $(x - 3)^3$ term in the Taylor series for f', the derivative of f, about $x = 3$?