Solutions

Practice

centered at $x=0$

1. The third Maclaurin polynomial for $\sin x$ is given by $p(x) = x - \frac{x^3}{3!}$. If this polynomial is used to approximate $\sin(0.1)$, what is the Lagrange error bound?

Interval: $[0, 0.1]$

$$\frac{\max\left[f^{(4)}(z)\right]\cdot(0.1-0)^4}{4!}$$

$f^4 = \sin x$

max of $\sin x = 1$

$$\frac{1\cdot(0.1)^4}{4!} = \boxed{4.1667 \times 10^{-6}}$$

2. If the Taylor Polynomial for approximating $\cos x$ is given by $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, what is the upper bound for the error in the approximation of $\cos(0.3)$?

$$\frac{\max\left[f^5(z)\right](0.3-0)^5}{5!}$$

Interval: $[0, 0.3]$
$n=4$
centered at $x=0$

$f^5 = -\sin x$
max of $-\sin x = 1$

$$\frac{1(0.3)^5}{5!} = \boxed{2.025 \times 10^{-5}}$$

3. If the Taylor Polynomial about $x = 0$ for the approximation of e^x is given by $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$, what is the upper bound for the error in the approximation of e?

e^1 means interval $[0, 1]$
$n=5$

$f^6 = e^x$

$$\frac{\max\left[f^6(z)\right](1-0)^6}{6!}$$

max on interval $[0,1]$ is e

$$\frac{e\cdot(1)^6}{6!} = \boxed{0.00377}$$

4. Let f be a function that has derivatives of all orders for all real numbers and let $P_3(x)$ be the third-degree Taylor Polynomial for f about $x = 0$. $\left|f^{(n)}(x)\right| \leq \frac{n}{n+1}$, for $1 \leq n \leq 5$ and all values of x. Of the following, which is the smallest value of k for which the Lagrange error bound guarantees that $|f(1) - P_3(1)| \leq k$?

$R_3(x) \leq k$

$$\frac{f^{(4)}(x)\,(1-0)^4}{4!} \leq k \longrightarrow \frac{\frac{4}{5}\cdot 1}{4!}$$

(A) $\frac{5}{6}$

(B) $\frac{5}{6} * \frac{1}{5!}$

(C) $\frac{5}{6} * \frac{1}{4!}$

(D) $\frac{4}{5} * \frac{1}{4!}$

5. The function f has derivatives of all orders for all real numbers, $f^{(4)}(x) = e^{\cos x}$. If the third-degree Taylor Polynomial for f about $x = 0$ is used to approximate f on the interval $[0,1]$, what is the Lagrange error bound?

$$\frac{\max\left[f^4(z)\right](1-0)^4}{4!}$$

$$\frac{e^1\cdot 1}{4!} \approx \boxed{0.11326}$$

6. The Taylor series for a function f about $x = 3$ is given by $\sum_{n=0}^{\infty}(-1)^n \frac{3n+1}{2^n}(x-3)^n$ and converges to f for $0 \le x \le 5$. If the third-degree Taylor Polynomial for f about $x = 3$ is used to approximate $f\left(\frac{13}{4}\right)$, what is the alternating series error bound?

$$a_n = \frac{3n+1}{2^n}(x-3)^n \longrightarrow a_4 = \frac{13}{16}\left(\frac{13}{4}-3\right)^4$$

$$\boxed{|R_3| \le \frac{13}{16}\left(\frac{1}{4}\right)^4}$$

7. Let f be a polynomial function with nonzero coefficients such that $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$. $T_4(x)$ is the fourth-degree Taylor Polynomial for f about $x = c$ such that $T_4 = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 + b_4(x-c)^4$. Based on the Lagrange error bound, $f(x) - T_4(x)$ must equal which of the following?

$$|R_4(x)| = |f(x) - T_4(x)|$$

$= \text{the next term}$

(A) x (B) $(x-c)^5$ (C) $a_5(x-c)^5$ (D) $\frac{a_5(x-c)^5}{5!}$

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8. Let $P(x)$ be the sixth-degree Taylor Polynomial for a function f about $x = 0$. Information about the maximum of the absolute value of selected derivatives of f over the interval $0 \le x \le 1.5$ is given below.

$\max_{0\le x\le 1.5}|f^{(5)}(x)| = 9.3$ $\max_{0\le x\le 1.5}|f^{(6)}(x)| = 62.1$ $\max_{0\le x\le 1.5}|f^{(7)}(x)| = 481.3$

What is the smallest value of k for which the Lagrange error bound guarantees that $|f(1.5) - P(1.5)| \le k$?

$$\frac{\max[f^7(z)](1.5-0)^7}{7!} = \frac{(481.3)(1.5)^7}{7!} \approx \boxed{1.6316}$$

9. The function f has derivatives of all orders for all real numbers. Values of f and its first four derivatives at $x = 2$ are given in the table.

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$
2	6	-12	18	-24	34

a. Write the third-degree Taylor Polynomial for f about $x = 2$, and use it to approximate $f(1.5)$.

$$P_3(x) = 6 + -12(x-2) + \frac{18(x-2)^2}{2} + \frac{-24(x-2)^3}{3!}$$

$$P_3(x) = 6 - 12(x-2) + 9(x-2)^2 - 4(x-2)^3$$

$$f(1.5) \approx P_3(1.5) = \boxed{14.75}$$

b. The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \leq 48$ for all $x > 1$. Use the Lagrange error bound to show that the approximation found in part (a) differs from $f(1.5)$ by no more than $\frac{1}{8}$.

$$R_3(1.5) \leq \frac{\max\left[f^4(z)\right](1.5-2)^4}{4!}$$

$$\leq \frac{48(-0.5)^4}{4!}$$

$$\leq 0.125 = \frac{1}{8}$$

10. Let h be a function having derivatives of all orders for $x > 0$. Selected values for the first four derivatives of h are given for $x = 3$. Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 3$ approximates $h(2.9)$ with an error less than 3×10^{-4}.

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
3	317	$\frac{753}{4}$	$\frac{1383}{4}$	$\frac{3483}{8}$	$\frac{1125}{16}$

$$R_3(x) \leq \frac{\max\left[h^4(z)\right](x-3)^4}{4!}$$

$$\leq \left(\frac{1125}{16}\right)\left(\frac{1}{24}\right)(-0.1)^4$$

$$\leq 2.9297 \times 10^{-4} \checkmark$$

10.12 Lagrange Error Bound

Test Prep

11. **Calculator allowed.**

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$
3	4	-8	14	-22	30

The function f has derivatives of all orders for all real numbers. Values of f and its first four derivatives at $x = 3$ are given in the table.

a. Write an equation for the line tangent to the graph of f at $x = 3$ and use it to approximate $f(2.5)$.

$$y - 4 = -8(x-3)$$

$$y = -8x + 28$$

$$f(2.5) \approx -8(2.5) + 28 = 8$$

b. Write the third-degree Taylor polynomial for f about $x = 3$, and use it to approximate $f(2.5)$.

$$P_3(x) = 4 + -8(x-3) + \frac{14(x-3)^2}{2} + \frac{-22(x-3)^3}{3!}$$

$$P_3(x) = 4 - 8(x-3) + 7(x-3)^2 - \frac{11}{3}(x-3)^3$$

$$f(2.5) \approx P_3(2.5) = 10.208$$

c. Is there enough information to determine whether f has a critical point at $x = 2.5$? If not, explain why not. If so, determine whether $f(2.5)$ is a relative maximum, relative minimum, or neither, and give a reason for your answer.

There is not enough information. We don't know if $f'(2.5) = 0$ or if $f'(2.5)$ does not exist. The Taylor Polynomial only gives us an approximation of $f(x)$.

d. The fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right| \leq 48$ for all $x > 2$. Use the Lagrange error bound to show that the approximation found in part (b) differs from $f(2.5)$ by no more than $\frac{1}{8}$.

$$R_3(1.5) \leq \frac{\max\left[f^4(z)\right](2.5-3)^4}{4!}$$

$$\leq \frac{48(-0.5)^4}{4!}$$

$$\leq 0.125$$

e. What is the coefficient of the $(x-3)^3$ term in the Taylor series for f', the derivative of f, about $x = 3$?

In the Taylor Polynomial for f the coefficient of $(x-3)^4$ is $\frac{f^{(4)}(3)}{4!} = \frac{30}{24} = \frac{5}{4}$, therefore in the Taylor Polynomial for f', the coefficient of $(x-3)^3$ is $\frac{5}{4} * 4 = 5$