## 10.12 Lagrange Error Bound

Calculus

sin(0.1), what is the Lagrange error bound?  

$$\frac{m \sim x \left[ \varsigma^{(+)}(z) \right] \cdot (0.1 - 0)}{4!} \quad \text{Interval: [0, 0.1]}$$

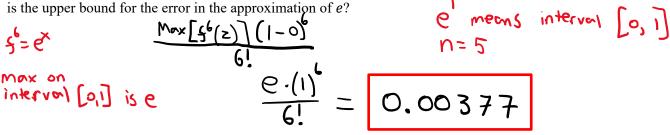
$$\frac{m \sim x \left[ \varsigma^{(+)}(z) \right] \cdot (0.1 - 0)}{4!} = \frac{1.1667}{4!} \times 10^{-6}$$

2. If the Taylor Polynomial for approximating  $\cos x$  is given by  $1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ , what is the upper bound for the error in the approximation of  $\cos(0.3)$ ?

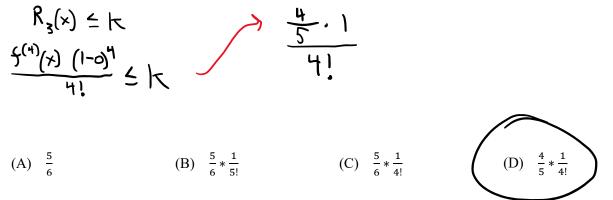
$$\max \left[ \frac{f^{5}(z)}{5!} \left( 0.3 - 0 \right) \right]$$

$$\sum_{\substack{n=4\\ (entered at x=0)}} \frac{1(0.3)^{5}}{5!} = 2.025 \times 10^{-5}$$

3. If the Taylor Polynomial about x = 0 for the approximation of  $e^x$  is given by  $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$ , what is the upper bound for the error in the approximation of e?



4. Let f be a function that has derivatives of all orders for all real numbers and let  $P_3(x)$  be the third-degree Taylor Polynomial for f about x = 0.  $|f^{(n)}(x)| \le \frac{n}{n+1}$ , for  $1 \le n \le 5$  and all values of x. Of the following, which is the smallest value of k for which the Lagrange error bound guarantees that  $|f(1) - P_3(1)| \le k$ ?



5. The function f has derivatives of all orders for all real numbers,  $f^{(4)}(x) = e^{\cos x}$ . If the third-degree Taylor Polynomial for f about x = 0 is used to approximate f on the interval [0,1], what is the Lagrange error bound?

$$\frac{e^{1} \cdot 1}{4!} \approx 0.11326$$

6. The Taylor series for a function f about x = 3 is given by  $\sum_{n=0}^{\infty} (-1)^n \frac{3n+1}{2^n} (x-3)^n$  and converges to f for  $0 \le x \le 5$ . If the third-degree Taylor Polynomial for f about x = 3 is used to approximate  $f\left(\frac{13}{4}\right)$ , what is the alternating series error bound?

$$a_{n} = \frac{3n+1}{2^{n}} (x-3)^{n} \longrightarrow a_{4} = \frac{13}{16} (\frac{13}{4} - 3)^{n}$$
$$|R_{3}| \leq \frac{13}{16} (\frac{1}{4})^{n}$$

7. Let f be a polynomial function with nonzero coefficients such that  $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$ .  $T_4(x)$  is the fourth-degree Taylor Polynomial for f about x = c such that  $T_4 = b_0 + b_1(x - c) + b_2(x - c)^2 + b_3(x - c)^3 + b_4(x - c)^4$ . Based on the Lagrange error bound,  $f(x) - T_4(x)$  must equal which of the following?

(A) x  
(B) 
$$(x-c)^5$$
  
(C)  $a_5(x-c)^5$   
(D)  $\frac{a_5(x-c)^5}{5!}$ 

8. Let P(x) be the sixth-degree Taylor Polynomial for a function f about x = 0. Information about the maximum of the absolute value of selected derivatives of f over the interval  $0 \le x \le 1.5$  is given below.

$$\max_{0 \le x \le 1.5} \left| f^{(5)}(x) \right| = 9.3 \qquad \max_{0 \le x \le 1.5} \left| f^{(6)}(x) \right| = 62.1 \qquad \max_{0 \le x \le 1.5} \left| f^{(7)}(x) \right| = 481.3$$

What is the smallest value of k for which the Lagrange error bound guarantees that  $|f(1.5) - P(1.5)| \le k$ ?

$$\frac{\max[s^{2}(z)](1.5-0)}{7!} = \frac{(481.3)(1.5)}{7!} \approx 1.6316$$

9. The function f has derivatives of all orders for all real numbers. Values of f and its first four derivatives at x = 2 are given in the table.

x	f(x)	f'(x)	$f^{\prime\prime}(x)$	$f^{\prime\prime\prime}(x)$	$f^{(4)}(x)$
2	6	-12	18	-24	34

a. Write the third-degree Taylor Polynomial for f about x = 2, and use it to approximate f(1.5).

$$P_{3}(x) = 6 + -12(x-2) + \frac{18(x-2)^{2}}{2} + \frac{-24(x-2)^{3}}{3!}$$

$$P_{3}(x) = 6 - 12(x-2) + 9(x-2)^{2} - 4(x-2)^{3}$$

$$f(1.5) \lesssim P_{3}(1.5) = 14.75$$

b. The fourth derivative of f satisfies the inequality  $|f^{(4)}(x)| \le 48$  for all x > 1. Use the Lagrange error bound to show that the approximation found in part (a) differs from f(1.5) by no more than  $\frac{1}{8}$ .

$$R_{3}(1.5) \leq \frac{m^{n} \times [f^{4}(z)](1.5-2)^{4}}{4!}$$

$$\leq \frac{48(-0.5)^{4}}{4!}$$

$$\leq 0.125 = \frac{1}{8}$$

10. Let *h* be a function having derivatives of all orders for x > 0. Selected values for the first four derivatives of *h* are given for x = 3. Use the Lagrange error bound to show that the third-degree Taylor polynomial for *h* about x = 3 approximates h(2.9) with an error less than  $3 \times 10^{-4}$ .

x	h(x)	h'(x)	<i>h</i> ′′( <i>x</i> )	<i>h'''</i> ( <i>x</i> )	$h^{(4)}(x)$	
3	317	$\frac{753}{4}$	$\frac{1383}{4}$	<u>3483</u> 8	$\frac{1125}{16}$	
$R_{3}(x) \stackrel{=}{=} \frac{n_{0} \times [h^{4}(z)](x-3)^{4}}{4!}$ $\stackrel{=}{=} \frac{(1125)(1-3)^{4}}{16}(-0.1)^{4}$ $\stackrel{=}{=} 2.9297 \times [0^{-4}] $						

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## **Test Prep**

11. Calculato<u>r allowed.</u>

x	f(x)	f'(x)	$f^{\prime\prime}(x)$	$f^{\prime\prime\prime}(x)$	$f^{(4)}(x)$
3	4	-8	14	-22	30

The function f has derivatives of all orders for all real numbers. Values of f and its first four derivatives at x = 3 are given in the table.

a. Write an equation for the line tangent to the graph of f at x = 3 and use it to approximate f(2.5).

$$y - 4 = -8(x - 3)$$
  
 $y = -8x + 28$   
 $f(2.5) = -8(2.5) + 28 = 8$ 

b. Write the third-degree Taylor polynomial for f about x = 3, and use it to approximate f(2.5).

$$P_{3}(x) = 4 + -8(x-3) + 14(x-3)^{2} + -\frac{22(x-3)^{3}}{3!} + \frac{3!}{3!} + \frac{3$$

c. Is there enough information to determine whether f has a critical point at x = 2.5? If not, explain why not. If so, determine whether f(2.5) is a relative maximum, relative minimum, or neither, and give a reason for your answer.

There is not enough information. We don't know if f'(2.5) = 0 or if f'(2.5) does not exist. The Taylor Polynomial only gives us an approximation of f(x).

d. The fourth derivative of f satisfies the inequality  $|f^{(4)}(x)| \le 48$  for all x > 2. Use the Lagrange error bound to show that the approximation found in part (b) differs from f(2.5) by no more than  $\frac{1}{8}$ .  $R (1.5) \ge m^{n} \ge \left[ \frac{1}{2} + \frac{1}{2} \right] (2.5-3)^{\frac{1}{2}}$ 

$$\begin{array}{r} R_{3}(1.5) \leq \frac{m^{o} \times \left[ \frac{5}{2} \right] (2.5-3)}{4!} \\ \leq \frac{49(-0.5)^{4}}{4!} \\ \leq 0.125 \end{array}$$

e. What is the coefficient of the  $(x - 3)^3$  term in the Taylor series for f', the derivative of f, about x = 3?

In the Taylor Polynomial for f the coefficient of  $(x - 3)^4$  is  $\frac{f^{(4)}(3)}{4!} = \frac{30}{24} = \frac{5}{4}$ , therefore in the Taylor Polynomial for f', the coefficient of  $(x - 3)^3$  is  $\frac{5}{4} * 4 = 5$