10.13 Radius and Interval of Convergence of Power Series

Calculus

Solutions

Practice

Find the interval of convergence for each power series.

1. $\sum_{n=0}^{\infty} \frac{(x-1)^n}{4^n}$ $\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{(x-1)^n} \right|$

$\lim_{n\to\infty} \left| \frac{(x-1)}{4} \right| < 1$

$-1 < \frac{x-1}{4} < 1$

$-4 < x-1 < 4$

$-3 < x < 5$

Check endpoints!

$x=-3$: $\sum_{n=0}^{\infty} \frac{(-4)^n}{4^n}$ diverges

$x=5$: $\sum_{n=0}^{\infty} \frac{4^n}{4^n} = 1$ diverges

$\boxed{-3 < x < 5}$

2. $\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^n}$ $\lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(x+2)^n} \right|$

$\lim_{n\to\infty} \left| \frac{(x+2)}{3} \right| < 1$

$-1 < \frac{x+2}{3} < 1$

$-3 < x+2 < 3$

$-5 < x < 1$

Check endpoints

$x=-5$: $\sum_{n=0}^{\infty} \frac{(-3)^n}{3^n}$ diverges

$x=1$: $\sum_{n=0}^{\infty} \frac{3^n}{3^n} = 1$ diverges

$\boxed{-5 < x < 1}$

3. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n}$ $\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n \cdot 2^n}{(x-2)^n} \right|$

$\lim_{n\to\infty} \left| \frac{(x-2)}{2} \cdot \underbrace{\frac{n}{n+1}}_{\text{approaches } 1} \right| < 1$

$-1 < \frac{x-2}{2} < 1$

$-2 < x-2 < 2$

$0 < x < 4$

Check endpoints

$x=0$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-2)^n}{n2^n}$ Diverges

$x=4$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cancel{(2)^n}}{n\cancel{2^n}}$ converges!

$\boxed{0 < x \leq 4}$

4. $\sum_{n=0}^{\infty} (2n)! \left(\frac{x}{3}\right)^n$

$\lim_{n\to\infty} \left| (2n+2)! \left(\frac{x}{3}\right)^{n+1} \cdot \frac{1}{(2n)!} \left(\frac{3}{x}\right)^n \right|$

$\lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)\cancel{(2n)!}}{\cancel{(2n)!}} \cdot \left(\frac{x}{3}\right) \right|$

∞

$\boxed{\text{Converges only at center } x=0.}$

Find the radius of convergence for each series.

5. $\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$

$\lim_{n\to\infty} \left| \frac{(4x)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(4x)^n} \right|$

$\lim_{n\to\infty} \left| \frac{n^2}{(n+1)^2} \cdot (4x) \right| < 1$

approaches 1

$-1 < 4x < 1$

$-\frac{1}{4} < x < \frac{1}{4}$

$\boxed{\text{radius} = \frac{1}{4}}$

6. $\sum_{n=0}^{\infty} \frac{(x-4)^{n+1}}{2 \cdot 3^{n+1}}$

$\lim_{n\to\infty} \left| \frac{(x-4)^{n+2}}{2 \cdot 3^{n+2}} \cdot \frac{2 \cdot 3^{n+1}}{(x-4)^{n+1}} \right|$

$\lim_{n\to\infty} \left| (x-4) \cdot \frac{1}{3} \right| < 1$

$-1 < \frac{x-4}{3} < 1$

$-3 < x-4 < 3$

$1 < x < 7$

$\boxed{\text{radius} = 3}$

7. $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

$\lim_{n\to\infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right|$

$\lim_{n\to\infty} \left| \frac{1}{(2n+2)(2n+1)} \cdot \frac{x^2}{1} \right| = 0$

converges for all values of x.

$\boxed{\text{Radius} = \infty}$

8. $\sum_{n=0}^{\infty} \frac{(2n)!\, x^{2n}}{n!}$

$\lim_{n\to\infty} \left| \frac{(2n+2)!\, x^{2n+2}}{(n+1)!} \cdot \frac{n!}{(2n)!\, x^{2n}} \right|$

$\lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)}{n+1} \cdot \frac{x^2}{1} \right| = \infty$

approaches ∞

converges to center.

$\boxed{\text{radius} = 0}$

What are all values of x for which each series converges?

9. $\sum_{n=1}^{\infty} \left(\frac{4}{x^2+1}\right)^n$

$\lim_{n\to\infty} \left| \left(\frac{4}{x^2+1}\right)^{n+1} \cdot \left(\frac{x^2+1}{4}\right)^n \right|$

$\lim_{n\to\infty} \left| \frac{4}{x^2+1} \right| < 1$

Only true if $x^2+1 > 4$

$x^2 > 3$

$x > \sqrt{3}$ or $x < -\sqrt{3}$

Both points lead to divergent series.

$\boxed{x > \sqrt{3} \text{ or } x < -\sqrt{3}}$

10. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(x + \frac{3}{2}\right)^n$

$\lim_{n\to\infty} \left| \frac{1}{n+1} \left(x+\frac{3}{2}\right)^{n+1} \cdot \frac{n}{(x+\frac{3}{2})^n} \right|$

$\lim_{n\to\infty} \left| \frac{n}{n+1} \cdot \left(x+\frac{3}{2}\right) \right| < 1$

$-1 < x + \frac{3}{2} < 1$

$-\frac{5}{2} < x < -\frac{1}{2}$

$x = -\frac{5}{2}$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (-1)^n$

$\frac{1}{n}$

diverges!

$\boxed{-\frac{5}{2} < x \leq -\frac{1}{2}}$

$x = -\frac{1}{2}$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} (1)^n$

converges

11. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 3^n}$

$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)\,3^{n+1}} \cdot \frac{n \cdot 3^n}{(x-2)^n} \right|$

$\lim_{n\to\infty} \left| \frac{n}{(n+1)} \cdot \frac{x-2}{3} \right| < 1$

$-1 < \frac{x-2}{3} < 1$

$-3 < x-2 < 3$

$-1 < x < 5$

$\underline{x=-1}$

$\sum_{n=1}^{\infty} \frac{(-3)^n}{n\,3^n}$

Converges

$\underline{x=5}$

$\sum_{n=1}^{\infty} \frac{3^n}{n\,3^n}$

diverges

$\boxed{-1 \le x < 5}$

12. $\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$

$\lim_{n\to\infty} \left| \frac{x^{5n+5}}{(n+1)!} \cdot \frac{n!}{x^{5n}} \right|$

$\lim_{n\to\infty} \left| \frac{1}{(n+1)} \cdot x^5 \right| = 0$

Converges for all values of x.

$\boxed{(-\infty, \infty)}$

10.13 Radius and Interval of Convergence of Power Series

Test Prep

13. The radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{(x-4)^{2n}}{n}$ is equal to 1. What is the interval of convergence?

Centered at $x=4$

$3 < x < 5$

$\boxed{3 < x < 5}$

Check endpoints

$\underline{x=3}$

$\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = \frac{1}{n}$

diverges

$\underline{x=5}$

$\sum_{n=1}^{\infty} \frac{1}{n}$

diverges

14. If the power series $\sum_{n=0}^{\infty} a_n (x-5)^n$ converges at $x = 8$ and diverges at $x = 10$, which of the following must be true?

Maybe I. The series converges at $x = 2$.

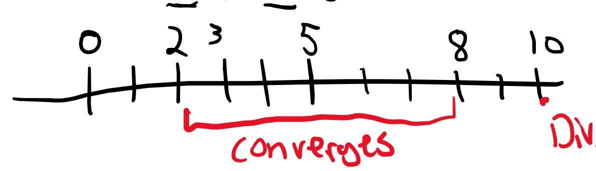
Yes II. The series converges at $x = 3$.

maybe III. The series diverges at $x = 0$.

center at $x=5$

radius is between 3 and 5

$3 \le r \le 5$



(A) I only

(B) II only

(C) I and II only

(D) II and III only

15. The coefficients of the power series $\sum_{n=0}^{\infty} a_n(x-3)^n$ satisfy $a_0 = 6$ and $a_n = \left(\frac{2n+1}{3n+1}\right) a_{n-1}$ for all $n \geq 1$. What is the radius of convergence?

$$\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = ?$$

$$b_n = a_n(x-3)^n$$

$$b_{n+1} = a_{n+1}(x-3)^{n+1}$$

$$a_{n+1} = \left(\frac{2(n+1)+1}{3(n+1)+1}\right)a_n$$

$$a_{n+1} = \frac{2n+3}{3n+4}a_n$$

$$\frac{a_{n+1}}{a_n} = \frac{2n+3}{3n+4}$$

$$\lim_{n\to\infty} \left|\frac{2n+3}{3n+4} \cdot \frac{(x-3)^{n+1}}{(x-3)^n}\right|$$

$$\lim_{n\to\infty} \left|\frac{2n+3}{3n+4} \cdot (x-3)\right| < 1$$

$$-1 < \tfrac{2}{3}(x-3) < 1$$

$$-\tfrac{3}{2} < x-3 < \tfrac{3}{2}$$

$$\tfrac{3}{2} < x < \tfrac{9}{2}$$

Interval = 3 $\quad \boxed{\text{radius} = \tfrac{3}{2}}$

16. The radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$ is 5, what is the interval of convergence?

centered at $x=5$
radius $=5$

$0 < x < 10$

Endpoints

$x=0$	$x=10$
diverges	converges

(A) $-5 < x < 5$ (B) $-5 < x \leq 5$ (C) $0 < x < 10$ (D) $0 < x \leq 10$

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17. Let $a_n = \frac{1}{n\ln n}$ for $n \geq 3$ and let f be the function given by $f(x) = \frac{1}{x\ln x}$.

a. The function f is continuous, decreasing, and positive. Use the Integral Test to determine the convergence or divergence of the series $\sum_{n=3}^{\infty} a_n$.

$$\int_3^\infty \frac{1}{n\ln n}\,dn = \lim_{t\to\infty}\int_3^t \frac{1}{n\ln n}\,dn$$

Let $u = \ln n$
$du = \frac{1}{n}dn$
$n\,du = dn$

$$\lim_{t\to\infty}\int_3^t \frac{1}{u}\,du$$

$$\lim_{t\to\infty} \ln|u|\Big|_3^t$$

$$\infty - 3 = \infty$$

Diverges by the Integral Test.

b. Find the interval of convergence of the power series $\sum_{n=3}^{\infty} \frac{(x-2)^{n+1}}{n \ln n}$.

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+2}}{(n+1)\ln(n+1)} \cdot \frac{n \ln n}{(x-2)^{n+1}} \right|$$

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n}(x-2)^{2}}{(x-2)^{n}(x-2)^{1}} \cdot \frac{n \ln n}{(n+1)\ln(n+1)} \right|$$

1

$\frac{\infty}{\infty} \rightarrow$ use L'Hopital's Rule

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \frac{1}{n} \cdot \frac{n+1}{1} = 1$$

$$\lim_{n\to\infty} |x-2| < 1$$

$$-1 < x-2 < 1$$

$$1 < x < 3$$

*Check endpoints

$\underline{x=1}$

$$\sum_{n=3}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$$

converges

$\underline{x=3}$

$$\sum_{n=3}^{\infty} \frac{1}{n \ln(n)}$$

diverges

$$\boxed{1 \leq x < 3}$$