Solutions

Practice

1. What is the coefficient of x^6 in the Taylor Series about $x = 0$ for the function $f(x) = \frac{e^{2x^2}}{2}$?

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

x^6

$$\frac{e^{2x^2}}{2} = \frac{1}{2}\left[1 + 2x^2 + \frac{(2x^2)^2}{2!} + \frac{(2x^2)^3}{3!} + \cdots\right]$$

$$\frac{1}{2}\left[\frac{8x^6}{6}\right] = \frac{2}{3}x^6$$

$$\boxed{2/3}$$

2. If $f(x) = x \sin 3x$, what is the Taylor Series for f about $x = 0$? Write the first four non-zero terms.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$x \sin(3x) = x\left[3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!}\right]$$

$$= \boxed{3x^2 - \frac{27x^4}{3!} + \frac{243x^6}{5!} - \frac{2187x^8}{7!}}$$

3. What is the Maclaurin Series for $\frac{1}{(1-x)^2}$? Write the first four non-zero terms.

$$\frac{1}{(1-x)} \cdot \frac{1}{(1-x)}$$

$$\sum_{n=0}^{\infty} x^n \cdot \sum_{n=0}^{\infty} x^n$$

$$[1 + x + x^2 + x^3 + \cdots][1 + x + x^2 + x^3 + \cdots]$$

$$\begin{array}{l} 1 + x + x^2 + x^3 + \cdots \\ \quad + x + x^2 + x^3 + x^4 + \cdots \\ \qquad + x^2 + x^3 + x^4 + x^5 \\ + \qquad + x^3 + x^4 + x^5 + x^6 \\ \hline \end{array}$$

$$\boxed{1 + 2x + 3x^2 + 4x^3}$$

4. What is the Maclaurin Series for the function $f(x) = \frac{1}{2}(e^x + e^{-x})$? Write the first four non-zero terms.

$$\frac{1}{2}e^x = \frac{1}{2} + \frac{1}{2}x + \frac{1}{2}\left(\frac{x^2}{2}\right) + \frac{1}{2}\left(\frac{x^3}{3!}\right) + \frac{1}{2}\left(\frac{x^4}{4!}\right) + \frac{1}{2}\left(\frac{x^5}{5!}\right) + \frac{1}{2}\left(\frac{x^6}{6!}\right)$$

$$\frac{1}{2}e^{-x} = \frac{1}{2} + \frac{1}{2}(-x) + \frac{1}{2}\left(\frac{x^2}{2}\right) + \frac{1}{2}\left(-\frac{x^3}{3!}\right) + \frac{1}{2}\left(\frac{x^4}{4!}\right) + \frac{1}{2}\left(-\frac{x^5}{5!}\right) + \frac{1}{2}\left(\frac{x^6}{6!}\right)$$

add

$$1 + 0 + \frac{x^2}{2} + 0 + \frac{x^4}{4!} + 0 + \frac{x^6}{6!}$$

$$\boxed{1 + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^6}{6!}}$$

5. Find the Maclaurin Series for the function $f(x) = \cos\sqrt{x}$. Write the first four non-zero terms.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\boxed{\cos\sqrt{x} = 1 - \frac{x}{2} + \frac{x^2}{4!} - \frac{x^3}{6!}}$$

6. Find the Maclaurin Series for the function $f(x) = \sin 5x$. Write the first four non-zero terms.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\boxed{\sin 5x = 5x - \frac{125x^3}{3!} + \frac{5^5 x^5}{5!} - \frac{5^7 x^7}{7!}}$$

7. What is the Taylor series expansion about $x = 0$ for the function $f(x) = \frac{\sin x}{x}$? Write the first four non-zero terms.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\boxed{\frac{1}{x} \cdot \sin x = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}}$$

8. The sum of the series $1 + \frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \cdots + \frac{3^n}{n!}$ is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad e^3 = \sum_{n=0}^{\infty} \frac{3^n}{n!}$$

(A) $\ln 3$ (B) e^3 (C) $\cos 3$ (D) $\sin 3$

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9. What is the sum of the series $\underbrace{1 + \ln 3 + \frac{(\ln 3)^2}{2!} + \cdots + \frac{(\ln 3)^n}{n!}}_{\text{follows pattern of } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ where } x = \ln 3}$?

$$e^{\ln 3} = \boxed{3}$$

10. What is the Taylor Series about $x = 0$ for the function $f(x) = 1 + x^2 + \cos x$? Write the first four non-zero terms.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$1 + x^2 + \cos x = \boxed{2 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}}$$

11. What is the sum of the infinite series $1-\left(\frac{\pi}{2}\right)^2\left(\frac{1}{3!}\right)+\left(\frac{\pi}{2}\right)^4\left(\frac{1}{5!}\right)-\left(\frac{\pi}{2}\right)^6\left(\frac{1}{7!}\right)+\cdots+\frac{\left(\frac{\pi}{2}\right)^{2n}(-1)^n}{(2n+1)!}$?

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. Dividing by x gives $\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$

Let $x = \pi/2$ $\quad \frac{\sin \pi/2}{\pi/2} = \frac{1}{\pi/2} = \boxed{2/\pi}$

12. Find the Maclaurin Series for the function $f(x) = e^{-3x}$. Write the first four non-zero terms.

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$ could reduce

$\boxed{e^{-3x} = 1 - 3x + \frac{9x^2}{2} - \frac{27x^3}{6}}$

13. Find the Maclaurin Series for the function $f(x) = \frac{\sin x^2}{x} + \cos x$. Write the first four non-zero terms.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ $\qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

$\frac{\sin x^2}{x} = \frac{x^2}{x} - \frac{(x^2)^3}{x\,3!} + \frac{(x^2)^5}{x\,5!} - \frac{(x^2)^7}{x\cdot 7!}$

$x - \frac{x^5}{3!} + \frac{x^9}{5!} - \frac{x^{13}}{7!}$

add and keep first four terms.

$\boxed{1 + x - \frac{x^2}{2} + \frac{x^4}{4!}}$

14. Which of the following is the Maclaurin Series for the function $f(x) = x\cos 2x$?

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$x\cos(2x) = x - \frac{x(2x)^2}{2} + \frac{x(2x)^4}{4!} - \frac{x(2x)^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} \cdot (2)^{2n} \cdot x}{(2n)!}$

(A) $\sum_{n=0}^{\infty} \frac{(-1)^n 2x^{2n}}{(2n)!}$ (B) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$ (C) $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n+1}}{(2n)!}$ (D) $\sum_{n=0}^{\infty} \frac{(-1)^n 2x^{2n+1}}{(2n)!}$

15. The Maclaurin series $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}$ represents which function $f(x)$

(A) $\sin x = x - \frac{x^3}{3!} + \cdots$ no!

(B) $-\sin x = -x + \frac{x^3}{3!} - \frac{x^5}{5!} \cdots$ no!

(D) $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \cdots$ Close!

(C) $\frac{1}{2}$ of choice (D) is correct.

(A) $\sin x$ (B) $-\sin x$ (C) $\frac{1}{2}(e^x - e^{-x})$ (D) $e^x - e^{-x}$

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16. The function f satisfies the equation $f'(x) = f(x) + x + 1$ and $f(0) = 2$. The Taylor Series for f about $x = 0$ converges to $f(x)$ for all x.

a. Write an equation for the line tangent to the curve of $y = f(x)$ at $x = 0$.

$f'(0) = f(0) + 0 + 1 = 2 + 1 = 3$

$y - 2 = 3(x - 0)$

$\boxed{y = 3x + 2}$

b. Find $f''(0)$ and find the second-degree Taylor Polynomial for f about $x = 0$.

$f''(0) = f'(0) + 1 = 3 + 1 = \boxed{4}$

$P_2(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2}$

$\boxed{P_2(x) = 2 + 3x + 2x^2}$

c. Find the fourth-degree Taylor Polynomial for f about $x = 0$.

$f'''(0) = f''(0) = 4$

$f^{4}(0) = f'''(0) = 4$

$$P_4(x) = 2 + 3x + 2x^2 + \frac{4x^3}{3!} + \frac{4x^4}{4!}$$

$$\boxed{P_4(x) = 2 + 3x + 2x^2 + \frac{2x^3}{3} + \frac{x^4}{6}}$$

d. Find $f^{(n)}(0)$, the n^{th} derivative of f about $x = 0$, for $n \geq 2$. Use the Taylor Series for f about $x = 0$ and the Taylor Series for e^x about $x = 0$ to find $f(x) - 4e^x$.

$f^{(n)}(0) = 4$ for $n \geq 2$

$$f(x) = 2 + 3x + 2x^2 + \frac{4x^3}{3!} + \frac{4x^4}{4!} + \cdots$$

$$4e^x = 4\left[1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right]$$

$$= 4 + 4x + 2x^2 + \frac{4x^3}{3!} + \frac{4x^4}{4!} + \cdots$$

$$f(x) - 4e^x = \boxed{-2 - x}$$