Solutions

Practice

1. What is the coefficient of x^2 in the Taylor Series for the function $f(x) = \sin^2 x$ about $x = 0$?

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\sin^2 x = \left(x - \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \cdots\right) = x^2 + \cdots$$

$$\boxed{1}$$

2. If the function f is defined as $f(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$, then what is $f'(x)$? Write the first four nonzero terms and the general term.

$$\boxed{f'(x) = \sum_{n=1}^{\infty} \frac{2n x^{2n-1}}{n!}} = \frac{2x^1}{1} + \frac{4x^3}{2!} + \frac{6x^5}{3!} + \frac{8x^7}{4!}$$

$$\boxed{2x + 2x^3 + x^5 + \frac{1}{3}x^7}$$

3. Use the power series expansion for $\cos x^6$ to evaluate the integral $\int_0^x \cos t^6 \, dt$. Write the first four nonzero terms and the general term.

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos t^6 = 1 - \frac{t^{12}}{2} + \frac{t^{24}}{4!} - \frac{t^{36}}{6!} + \cdots \frac{(-1)^n t^{12n}}{(2n)!}$$

$$\boxed{\int_0^x \cos t^6 = x - \frac{x^{13}}{26} + \frac{x^{25}}{25 \cdot 4!} - \frac{x^{37}}{37 \cdot 6!} + \cdots + \frac{(-1)^n x^{12n+1}}{(12n+1)(2n)!}}$$

4. For $x > 0$, the power series defined by $1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^n x^{2n}}{(2n+1)!}$ converges to which of the following?

$$\frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$

~~(A)~~ $\cos x$ even $\frac{x^2}{2!}$

~~(B)~~ $\sin x$ odd $\frac{x^3}{3!}$

(C) $\frac{\sin x}{x}$

(D) $e^x - e^{x^2}$

5. It is known that the Maclaurin series for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$. Use this fact to assist in finding the first four nonzero terms and the general term for the power series expansion for the function $\frac{x^2}{1-x^2}$.

$$\frac{1}{1-x} = x^n = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots + x^{2n}$$

$$\boxed{x^2 \cdot \frac{1}{1-x^2} = x^2 + x^4 + x^6 + x^8 + \cdots + x^{2n+2}}$$

6. Let f be the function with initial condition $f(0) = 0$ and derivative $f'(x) = \frac{1}{1+x^7}$. Write the first four nonzero terms of the Maclaurin series for the function f.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

$$\frac{1}{1+x^7} = 1 - x^7 + x^{14} - x^{21} + \cdots$$

$$\boxed{\int \frac{1}{1+x^7}\,dx = x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22}}$$

7. Find the Maclaurin series for the function $f(x) = e^{3x}$. Write the first four nonzero terms and the general term.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

$$\boxed{e^{3x} = 1 + 3x + \frac{9x^2}{2} + \frac{27x^3}{3!} + \cdots + \frac{(3x)^n}{n!}}$$

8. If a function has the derivative $f'(x) = \sin(x^2)$ and initial conditions $f(0) = 0$, write the first four nonzero terms of the Maclaurin series for f.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$$

$$f(x) = \int \sin(x^2)\,dx = \boxed{\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!}}$$

9. The function f has derivatives of all orders and the Maclaurin series for the function f is given by $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+3}$. Find the Maclaurin series for the derivative $f'(x)$. Write the first four nonzero terms and the general term.

$$f = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \frac{x^7}{9} + \cdots$$

$$\boxed{f' = \frac{1}{3} - \frac{3x^2}{5} + \frac{5x^4}{7} - \frac{7x^6}{9} + \cdots + \frac{(-1)^n (2n+1)\, x^{2n}}{2n+3}}$$

10. Let the function f be defined by $f(x) = \frac{1}{1-x}$. Find the Maclaurin series for the derivative f'. Write the first four nonzero terms and the general term.

$$f = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots x^n$$

$$\boxed{f' = 1 + 2x + 3x^2 + 4x^2 + \cdots + n x^{n-1}}$$

11. Find the second-degree Taylor Polynomial for the function $f(x) = \frac{\cos x}{1-x}$ about $x = 0$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$(\cos x)\left(\frac{1}{1-x}\right) = \left(1 - \frac{x^2}{2!}\right)(1 + x + x^2)$$

2nd degree!

$$1 + x + x^2 - \frac{x^2}{2} - \cancel{\frac{x^3}{2}} - \cancel{\frac{x^4}{2}}$$

$$\boxed{1 + x + \frac{x^2}{2}}$$

12. What is the coefficient of x^2 in the Maclaurin series for the function $f(x) = \left(\frac{1}{1+x}\right)^2$?

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

$$\left(\frac{1}{1+x}\right)^2 = (1 - x + x^2 \cdots)(1 - x + x^2 \cdots)$$

$$1 - x + x^2 - x + x^2 - x^3 + x^2 - x^3 + x^4 \cdots$$

$$\boxed{3}x^2$$

13. Find the Maclaurin series for the function $f(x) = x\cos x^2$. Write the first four nonzero terms and the general term.

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos x^2 = 1 - \frac{x^4}{2} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots + \frac{(-1)^n x^{4n}}{(2n)!}$$

$$\boxed{x\cos x^2 = x - \frac{x^5}{2} + \frac{x^9}{4!} - \frac{x^{13}}{6!} + \cdots + \frac{(-1)^n x^{4n+1}}{(2n)!}}$$

14. Given that f is a function that has derivatives of all orders and $f(1) = 3, f'(1) = -2, f''(1) = 2$, and $f'''(1) = 4$. Write the second-degree Taylor Polynomial for the derivative f' about $x = 1$ and use it to find the approximate value of $f'(1.2)$.

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2} + \frac{f'''(1)(x-1)^3}{3!}$$

$$P_3(x) = 3 - 2(x-1) + (x-1)^2 + \frac{4}{3!}(x-1)^3$$

$$P_2'(x) = -2 + 2(x-1) + 2(x-1)^2 \leftarrow 2^{nd} \text{ degree}$$

$$P_2'(1.2) = \boxed{-1.52}$$

15. Let the fourth-degree Taylor Polynomial be defined by $T = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3 + 6(x-4)^4$ for the function f about $x = 4$. Find the third-degree Taylor Polynomial for f' about $x = 4$ and then use it to approximate $f'(4.2)$.

$$T_3'(x) = -3 + 10(x-4) - 6(x-4)^2 + 24(x-4)^3$$

$$T_3'(4.2) \simeq \boxed{-1.048}$$

10.15 Representing Functions as Power Series

Test Prep

16. Given a function defined by $f(x) = \frac{\cos(2x)-1}{x^2}$ for $x \neq 0$ and is continuous for all real numbers x.

a. What is the limit of the function $f(x)$ as x approaches 0?

$$\frac{0}{0} \xrightarrow{\text{L'hopitals}} \lim_{x\to 0} \frac{-2\sin(2x)}{2x} = \frac{0}{0} \xrightarrow{\text{L'hopitals}} \lim_{x\to 0} \frac{-4\cos(2x)}{2} = \boxed{-2}$$

b. Write the first four nonzero terms and the general term of the power series that represents the function $h(x) = \cos 2x$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\boxed{\cos(2x) = 1 - \frac{4x^2}{2} + \frac{16x^4}{4!} - \frac{64x^6}{6!} + \cdots \frac{(-1)^n 4^n x^{2n}}{(2n)!}}$$

c. Use the results from part (b) to write the first three nonzero terms for $f(x) = \frac{\cos(2x)-1}{x^2}$.

$$\cos(2x) - 1 = -2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6$$

$$\frac{\cos(2x)-1}{x^2} = \boxed{-2 + \frac{2}{3}x^2 - \frac{4}{45}x^4}$$

d. Use the results from part (c) to determine if the function $f(x) = \frac{\cos(2x)-1}{x^2}$ has a relative maximum, a relative minimum or neither at $x = 0$. Justify your answer.

$$f' \approx \frac{4}{3}x - \frac{16}{45}x^3$$

$$x\left(\frac{4}{3} - \frac{16}{45}x^2\right) = 0$$

✓ $x = 0$ is a critical point

$$f'' \approx \frac{4}{3} - \frac{48}{45}x^2$$

$$f''(0) = \frac{4}{3} > 0$$

concave up

$x=0$ is a relative minimum because $f'(0) = 0$ and $f''(0) > 0$.