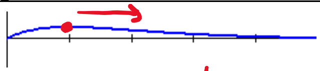
10.4 Integral Test for Convergence
Calculus

Solutions

Practice

If the Integral Test applies, use it to determine whether the series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$



pos, cont, decreasing

$\displaystyle\lim_{b\to\infty}\int_1^b ne^{-n}\,dn$ → integration by parts

$f=n \quad g'=e^{-n}dn$

$f'=1 \quad g=-e^{-n}$

$-ne^{-n}+\int_1^b -e^{-n}\,dn$

$\displaystyle\lim_{b\to\infty}\left[-ne^{-n}-e^{-n}\right]\Big|_1^b$

$\displaystyle\lim_{b\to\infty}\left[-\frac{b}{e^b}-\frac{1}{e^b}\right]-\left[-\frac{1}{e}-\frac{1}{e}\right]$

$0+0-\left[-\frac{2}{e}\right]$

$2/e$

$\boxed{\text{Converges}}$

2. $\displaystyle\sum_{n=1}^{\infty} f(n) = 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots$

pos. ✓
cont. ✓
Dec. ✓

$a_n=\dfrac{1}{n^2}$

$\displaystyle\lim_{b\to\infty}\int_1^b n^{-2}\,dn$

$\displaystyle\lim_{b\to\infty} -\frac{1}{n}\Big|_1^b$

$\displaystyle\lim_{b\to\infty}\left[-\frac{1}{b}--1\right]$

$0+1$

1

$\boxed{\text{Converges}}$

3. $\displaystyle\sum_{n=1}^{\infty} f(n) = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

pos. ✓
cont. ✓
Dec. ✓

$a_n=\dfrac{1}{n}$

$\displaystyle\lim_{b\to\infty}\int_1^b \frac{1}{n}\,dn$

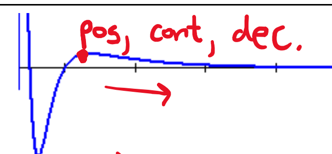
$\displaystyle\lim_{b\to\infty}\left[\ln n\right]\Big|_1^b$

$\displaystyle\lim_{b\to\infty}\left[\ln b-\ln 1\right]$

$\infty-0$

$\boxed{\text{Diverges}}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}\sin\frac{\pi}{n}$



pos, cont, dec.

$\displaystyle\lim_{b\to\infty}\int_1^b \frac{1}{n^2}\sin\frac{\pi}{n}\,dn$

$u=\frac{\pi}{n}$

$du=-\frac{\pi}{n^2}\,dn$

$-\frac{n^2}{\pi}\,du=dn$

$\displaystyle\lim_{b\to\infty}\int_1^b -\frac{1}{\pi}\sin u\,du$

$\displaystyle\lim_{b\to\infty}\frac{1}{\pi}\cos\left(\frac{\pi}{n}\right)\Big|_1^b$

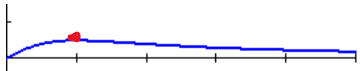
$\displaystyle\lim_{b\to\infty}\frac{1}{\pi}\left[\cos\left(\frac{\pi}{b}\right)-\cos(\pi)\right]$

$\frac{1}{\pi}\left[\cos(0)-(-1)\right]$

$\dfrac{2}{\pi}$

$\boxed{\text{Converges}}$

5. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$



pos, cont, dec.

$\lim_{b\to\infty} \int_1^b \frac{n}{n^2+1}\,dn$ $u = n^2+1$

$du = 2n\,dn$

$\lim_{b\to\infty} \int_1^b \frac{1}{2}\frac{1}{u}\,du$ $\frac{du}{2n} = dn$

$\lim_{b\to\infty} \left[\frac{1}{2}\ln|n^2+1|\right]\Big|_1^b$

$\lim_{b\to\infty} \left[\frac{1}{2}\ln|b^2+1| - \frac{1}{2}\ln 2\right]$

$\infty - \frac{1}{2}\ln 2$

$\boxed{\text{Diverges}}$

6. $\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k+2}$, where k is a positive integer. Assume the series meets the criteria for the Integral Test.

$\lim_{b\to\infty} \int_1^b \frac{1}{ku}\,du$ $u = n^k+2$

$du = kn^{k-1}dn$

$\lim_{b\to\infty} \frac{1}{k}\ln|n^k+2|\Big|_1^b$ $\frac{du}{kn^{k-1}} = dn$

$\lim_{b\to\infty} \frac{1}{k}\left[\ln|b^k+2| - \ln|1^k+2|\right]$

$\frac{1}{k}\left[\infty - \ln 3\right]$

$\boxed{\text{Diverges}}$

7. Let f be a positive, continuous, and decreasing function. If $\int_1^{\infty} f(x)\,dx = 4$, which of the following statements about the series $\sum_{n=1}^{\infty} f(n)$ must be true?

n=1 means Left rectangular approx. Overestimate

A. $\sum_{n=1}^{\infty} f(n) = 0$

B. $\sum_{n=1}^{\infty} f(n)$ converges, and $\sum_{n=1}^{\infty} f(n) > 4$

C. $\sum_{n=1}^{\infty} f(n)$ converges, and $\sum_{n=1}^{\infty} f(n) < 4$

D. $\sum_{n=1}^{\infty} f(n)$ diverges, and $\sum_{n=1}^{\infty} f(n) = 0$

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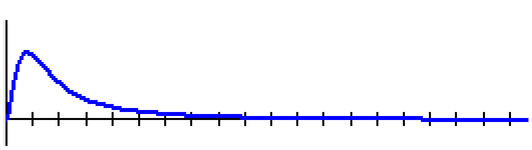
8. Explain why the Integral Test does not apply for the series $\sum_{x=1}^{\infty} e^x \sin x$.

f(x) is continuous, but not pos. and decr. for $x \geq 1$.

9. Show that the series $\sum_{x=1}^{\infty} \frac{\tan^{-1} x}{x^2+1}$ meets the criteria to apply the Integral Test for convergence.



Continuous for $x \geq 1$

Positive for $x \geq 1$

decreasing for $x \geq 1$

10. Let f be positive, continuous, and decreasing on $[1, \infty)$, such that $a_n = f(n)$. If $\sum_{n=1}^{\infty} a_n = 7$, which of the following must be true?

A. $\lim_{n\to\infty} a_n = 7$

B. $\int_1^{\infty} f(x)\,dx = 7$

C. $\int_1^{\infty} f(x)\,dx$ diverges

D. $\int_1^{\infty} f(x)\,dx$ converges

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11. Which of the following can be used to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n+3}{n+4}$?

X I. Properties of Geometric Series – not geometric

✓ II. nth-Term Test – $\lim_{n\to\infty} \frac{n+3}{n+4} = 1 \neq 0$ ✓

X III. Integral Test – increasing

A. I only

B. II only

C. III only

D. II and III only

E. I, II, and III

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12. Which of the following can be used to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$?

✓ I. Properties of Geometric Series $\left(\frac{1}{2}\right)^n$

X II. nth-Term Test $\rightarrow \lim_{n\to\infty} a_n = 0$

✓ III. Integral Test

pos, cont, dec.

A. I only

B. II only

C. III only

D. I and II only

E. I and III only

Test Prep

10.4 Integral Test for Convergence

13. Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^3}$. The integral test can be used to determine convergence or divergence of the series because $f(x) = \frac{1}{x^3}$ is positive, continuous, and decreasing on $[1, \infty)$. Which of the following is true?

A. $1 + \int_1^{\infty} \frac{1}{x^3}dx < \sum_{n=1}^{\infty} \frac{1}{n^3} < \int_1^{\infty} \frac{1}{x^3}dx$

B. $\int_1^{\infty} \frac{1}{x^3}dx < \sum_{n=1}^{\infty} \frac{1}{n^3} < 1 + \int_1^{\infty} \frac{1}{x^3}dx$

C. $\sum_{n=1}^{\infty} \frac{1}{n^3} < \int_1^{\infty} \frac{1}{x^3}dx < 1 + \int_1^{\infty} \frac{1}{x^3}dx$

D. $\int_1^{\infty} \frac{1}{x^3}dx < 1 + \int_1^{\infty} \frac{1}{x^3}dx < \sum_{n=1}^{\infty} \frac{1}{n^3}$