10.6 Comparison Tests for Convergence
Calculus

Solutions

Practice

1. Which of the following statements about convergence of the series $\sum_{n=1}^{\infty} \frac{1}{\ln(n+2)}$ is true?

(A) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+2)}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$

(B) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+2)}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$

(C) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+2)}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$

(D) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+2)}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$

Harmonic, diverges

$0 < \frac{1}{n} < \frac{1}{\ln(n+2)}$ ← must diverge

$0 < \frac{1}{\sqrt{n^3}} < \frac{1}{\ln(n+2)}$ ← unknown ?

converges

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2. Which of the following series converges?

(A) $\sum_{n=1}^{\infty} \frac{3n}{n^3+2}$ Compare to $\frac{1}{n^2}$, which converges

$\lim_{n\to\infty} \frac{3n}{n^3+2} \cdot \frac{n^2}{1} = \frac{3n^3}{n^3+2} = 3$ ✓ pos & Finite

Both converge

(B) $\sum_{n=1}^{\infty} \frac{5n}{2n+1}$ Diverges by n^{th} term test

$\lim_{n\to\infty} \frac{5n}{2n+1} = \frac{5}{2}$

(C) $\sum_{n=1}^{\infty} \frac{7n}{n^2+1}$ Compare to $\frac{1}{n}$ (diverges)

$\lim_{n\to\infty} \frac{7n}{n^2+1} \cdot \frac{n}{1} = \frac{7n^2}{n^2+1} = 7$

Both diverge

(D) $\sum_{n=1}^{\infty} \frac{5^n}{4^n+1}$ Compare to $\left(\frac{5}{4}\right)^n$ diverges

$\lim_{n\to\infty} \frac{5^n}{4^n+1} \cdot \frac{4^n}{5^n} = 1$

Both diverge

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3. Use the Comparison Test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2+5^n}$ converges or diverges.

Compare to $\frac{1}{5^n}$, which converges (geometric)

$0 < \frac{1}{2+5^n} < \frac{1}{5^n}$

Both converge by Comparison Test

4. Which of the following series can be used with the Limit Comparison Test to determine convergence of the series $\sum_{n=1}^{\infty} \frac{n^3}{n^4+3}$? "reduce" to $\frac{1}{n}$ for comparison. $\frac{1}{n}$ diverges

(A) $\sum_{n=1}^{\infty} \frac{n}{n+3}$

(B) $\sum_{n=1}^{\infty} \frac{1}{n^3+3}$

(C) $\sum_{n=1}^{\infty} \frac{1}{n}$

(D) $\sum_{n=1}^{\infty} \frac{1}{n^4}$

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5. Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n > 0$ and $b_n > 0$ for $n \geq 1$. If $\sum_{n=1}^{\infty} a_n$ diverges which of the following must be true?

If $0 < a_n \leq b_n$ then $\sum b_n$ diverges
If $0 < b_n \leq a_n$ then we have no conclusion.

(A) If $a_n \leq b_n$, then $\sum_{n=1}^{\infty} b_n$ converges.

(B) If $a_n \leq b_n$, then $\sum_{n=1}^{\infty} b_n$ diverges.

(C) If $b_n \leq a_n$, then $\sum_{n=1}^{\infty} b_n$ converges.

(D) If $b_n \leq a_n$, then $\sum_{n=1}^{\infty} b_n$ diverges.

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6. Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n > 0$ and $b_n > 0$ for $n \geq 1$. If $\sum_{n=1}^{\infty} b_n$ converges which of the following must be true?

If $b_n \leq a_n$, then no conclusion
If $b_n \geq a_n$, then $\sum a_n$ converges

(A) If $a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(B) If $a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ converges.

(C) If $b_n \leq a_n$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(D) If $b_n \leq a_n$, then $\sum_{n=1}^{\infty} a_n$ converges.

7. Let $a > 0, b > 0$, and $c > 0$. Determine whether the series $\sum_{n=0}^{\infty} \frac{1}{an^2 + bn + c}$ converges or diverges.

Compare to $\frac{1}{n^2}$ (converges)

$$\lim_{n\to\infty} \frac{1}{an^2+bn+c} \cdot \frac{n^2}{1} = \frac{1}{a}$$

$\boxed{\frac{1}{a} > 0\text{, therefore both converge by the Limit Comparison Test.}}$

8. Determine the convergence or divergence of the series $\sum_{n=2}^{\infty} \frac{1}{6^n + 6}$.

compare to $\frac{1}{6^n}$ (geometric, converges)

$$\lim_{n\to\infty} \frac{1}{6^n+6} \cdot \frac{6^n}{1} = 1$$

$\boxed{\text{Both converge by the Limit Comparison Test.}}$

9. For the series $\sum_{n=1}^{\infty} \frac{n3^n}{2n^4 - 2}$, which of the following could be used with the Limit Comparison Test?

"reduce" largest terms

(A) $\sum_{n=1}^{\infty} \frac{1}{n^4}$

(B) $\sum_{n=1}^{\infty} \frac{3^n}{n^4}$

(C) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(D) $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

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10. Which of the following can be used with the Comparison Test to determine the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$?

p-series where $0 < p \le 1$

B

(A) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \frac{1}{2+\sqrt{n}}$ diverges ?

(B) $\sum_{n=1}^{\infty} \frac{1}{n} < \frac{1}{2+\sqrt{n}}$ diverges Must diverge!

(C) $\sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{1}{2+\sqrt{n}}$ converges ?

(D) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n < \frac{1}{2+\sqrt{n}}$ converges (Geometric) ?

11. Which of the following series diverge?

I. $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$ Compare to $\frac{1}{n^{2/3}}$

Divergent p-series

$\lim_{n\to\infty} \frac{\sqrt[3]{n}}{n} \cdot \frac{n^{2/3}}{1} = 1$ ✓

II. $\sum_{n=1}^{\infty} \frac{1}{n^3 - 27}$ Compare to $\frac{1}{n^3}$

convergent p-series

$\lim_{n\to\infty} \frac{1}{n^3-27} \cdot \frac{n^3}{1} = 1$

Both converge ✗

III. $\sum_{n=1}^{\infty} \frac{1}{4^n + 1}$ Compare to $\frac{1}{4^n}$

convergent geometric

$0 < \frac{1}{4^n+1} < \frac{1}{4^n}$

Converges! ✗

(A) I only

(B) II only

(C) I and II only

(D) I, II, and III

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12. Consider the series $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$, where $p \geq 0$. For what values of p is the series convergent?

Compare to $\frac{1}{n^p}$. For convergence, we need $\frac{1}{n^p}$ to converge.

$0 < \frac{1}{n^p \ln(n)} \leq \frac{1}{n^p}$ p-series!

$\boxed{p > 1}$

13. Determine whether the series $\sum_{n=1}^{\infty} \frac{n-3}{n^3}$ converges or diverges.

Compare to $\frac{1}{n^2}$, convergent p-series

$\lim_{n\to\infty} \frac{n-3}{n^3} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{n-3}{n} = 1$

$\boxed{\text{Both converge.}}$

14. Consider the series $1 + \frac{1}{5} + \frac{1}{9} + \frac{1}{13} + \cdots = \sum_{n=1}^{\infty} \frac{1}{4n-3}$. Use the Limit Comparison Test with the series $\sum_{n=1}^{\infty} \frac{1}{4n}$ to determine the convergence of the series.

$\frac{1}{4n}$

Divergent (Harmonic p-series)

$\lim_{n\to\infty} \frac{1}{4n-3} \cdot \frac{4n}{1} = \lim_{n\to\infty} \frac{4n}{4n-3} = 1$

$\boxed{\text{Both diverge.}}$

15. Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n > 0$ and $b_n > 0$ for $n \geq 1$. If $a_n \leq b_n$, then which of the following must be true?

If a_n diverges, b_n diverges

or

If b_n converges, a_n converges

(A) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ diverges.

(B) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

(C) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

(D) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ converges.

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16. Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n > 0$ and $b_n > 0$ for $n \geq 1$. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 2$, then which of the following must be true?

I. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ diverges.

II. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ converges.

III. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

IV. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

(A) I only

(B) II only

(C) III only

(D) IV only

(E) I and II only

(F) III and IV only