10.8 Ratio Test
Calculus

Solutions

Practice

Determine whether the following series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{(n+1)3^n}{n!}$

$a_{n+1} = \frac{(n+2)\,3^{n+1}}{(n+1)!}$

$\lim_{n\to\infty} \frac{(n+2)\,3^n \cdot 3^1}{(n+1)\cdot n!} \cdot \frac{n!}{(n+1)3^n}$

$\lim_{n\to\infty} \frac{(n+2)\cdot 3}{(n+1)^2} = 0 < 1$

$\boxed{\text{Converges}}$

2. $\sum_{n=1}^{\infty} \frac{n!}{5^n}$

$a_{n+1} = \frac{(n+1)!}{5^{n+1}}$

$\lim_{n\to\infty} \frac{(n+1)\,n!}{5^n \cdot 5^1} \cdot \frac{5^n}{n!}$

$\lim_{n\to\infty} \frac{n+1}{5} = \infty > 1$

$\boxed{\text{Diverges}}$

3. What are values of $x > 0$ for which the series $\sum_{n=1}^{\infty} \frac{n6^n}{x^n}$ converges?

$a_{n+1} = \frac{(n+1)\,6^{n+1}}{x^{n+1}}$

$\lim_{n\to\infty} \frac{(n+1)\,6^n \cdot 6^1}{x^n \cdot x^1} \cdot \frac{x^n}{n \cdot 6^n} < 1$

$\lim_{n\to\infty} \frac{(n+1)\cdot 6}{nx} < 1$

$\lim_{n\to\infty} \frac{n+1}{n} = 1$, so we need $\frac{6}{x} < 1$

$6 < x$

$\boxed{x > 6}$

4. What are all positive values of p for which the series $\sum_{n=1}^{\infty} \frac{n^p}{7^n}$ will converge?

$a_{n+1} = \frac{(n+1)^p}{7^{n+1}}$

$\lim_{n\to\infty} \frac{(n+1)^p}{7^n \cdot 7^1} \cdot \frac{7^n}{n^p} < 1$

$\lim_{n\to\infty} \frac{(n+1)^p}{7 \cdot n^p} < 1$ $\quad \lim_{n\to\infty} \frac{1}{7} \cdot \left(\frac{n+1}{n}\right)^p < 1$

A. $p > 0$

B. $0 < p < 7$

C. $p > 1$

D. There are no positive values where the series will converge.

5. Which of the following series converge?

✓ I. $\sum_{n=1}^{\infty} \frac{7^n}{n!}$ ✗ II. $\sum_{n=1}^{\infty} \frac{n!}{n^{20}}$ ✓ III. $\sum_{n=1}^{\infty} \frac{\pi^{-2n}}{n}$

$$\lim_{n\to\infty} \frac{7^{n+1}}{(n+1)!} \cdot \frac{n!}{7^n}$$

$$\lim_{n\to\infty} \frac{\cancel{7^n}\cdot 7^1}{(n+1)\cdot \cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{7^n}} < 1$$

$$\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{20}} \cdot \frac{n^{20}}{n!}$$

$$\lim_{n\to\infty} \frac{(n+1)\cdot\cancel{n!}}{(n+1)^{20}} \cdot \frac{n^{20}}{\cancel{n!}} > 1$$

diverges

$$\lim_{n\to\infty} \frac{\pi^{-2n-2}}{n+1} \cdot \frac{n}{\pi^{-2n}}$$

$$\lim_{n\to\infty} \frac{1}{(n+1)\pi^{2n}\cdot\pi^2} \cdot n\pi^{2n}$$

$$\lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{1}{\pi^2} < 1$$

A. I only B. I and II only C. III only D. I and III only E. I, II, and III

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6. If the Ratio Test is applied to the series $\sum_{n=1}^{\infty} \frac{n\pi^n}{15^n}$, which of the following inequalities results, implying that the series converges?

$$a_{n+1} = \frac{(n+1)\cdot\pi^{n+1}}{15^{n+1}}$$

$$\lim_{n\to\infty} \frac{(n+1)\,\cancel{\pi^n}\cdot\pi}{\cancel{15^n}\cdot 15} \cdot \frac{\cancel{15^n}}{n\,\cancel{\pi^n}}$$

$$\lim_{n\to\infty} \frac{(n+1)\pi}{15n} = \frac{\pi}{15} < 1$$

A. $\lim_{n\to\infty} \frac{n\pi^n}{15^n} < 1$ B. $\lim_{n\to\infty} \frac{15^n}{n\pi^n} < 1$ C. $\lim_{n\to\infty} \frac{(n+1)\pi^{n+1}}{15^{n+1}} < 1$ D. $\lim_{n\to\infty} \frac{(n+1)\pi}{15n} < 1$

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7. If $a_n > 0$ for all n and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 6$, which of the following series converges?

> 1, diverges

$$\lim_{n\to\infty} \frac{a_{n+1}}{(n+1)^7} \cdot \frac{n^7}{a_n}$$

$$\lim_{n\to\infty} 6 \cdot \frac{n^7}{(n+1)^7}$$

> 1, diverges

$$\lim_{n\to\infty} \frac{a_{n+1}}{7^n\cdot 7^1} \cdot \frac{7^n}{a_n}$$

$$\lim_{n\to\infty} 6 \cdot \frac{1}{7} < 1$$

$$\lim_{n\to\infty} \frac{(a_{n+1})^2}{7^n\cdot 7^1} \cdot \frac{7^n}{(a_n)^2}$$

$$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right)^2 \cdot \frac{1}{7}$$

$$36 \cdot \frac{1}{7}$$

> 1, diverges

A. $\sum_{n=1}^{\infty} a_n$ B. $\sum_{n=1}^{\infty} \frac{a_n}{n^7}$ C. $\sum_{n=1}^{\infty} \frac{a_n}{7^n}$ D. $\sum_{n=1}^{\infty} \frac{(a_n)^2}{7^n}$

8. Consider the series $\sum_{n=1}^{\infty} \frac{n!}{3^n}$. If the Ratio Test is applied to the series, which of the following inequalities results, implying the series diverges?

$a_{n+1} = \frac{(n+1)!}{3^{n+1}}$

$\lim_{n\to\infty} \frac{(n+1)\cdot \cancel{n!}}{\cancel{3^n}\cdot 3^1} \cdot \frac{\cancel{3^n}}{\cancel{n!}}$

$\lim_{n\to\infty} \frac{n+1}{3} > 1$

A. $\lim_{n\to\infty} \frac{n!}{3^n} < 1$

B. $\lim_{n\to\infty} \frac{n!}{3^n} > 1$

C. $\lim_{n\to\infty} \frac{n+1}{3} < 1$

D. $\lim_{n\to\infty} \frac{n+1}{3} > 1$

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9. For which of the series is the Ratio Test inconclusive?

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$

I. $\sum_{n=1}^{\infty} \frac{1}{3n}$

$\lim_{n\to\infty} \frac{1}{3(n+1)} \cdot \frac{3n}{1} = \frac{3}{3}$

$= 1$

Inconclusive

II. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$

$\lim_{n\to\infty} \frac{\sqrt{n+1}}{n+2} \cdot \frac{n+1}{\sqrt{n}}$

$\approx \lim_{n\to\infty} \frac{\sqrt{n}\cdot n}{n\cdot\sqrt{n}} = 1$

Inconclusive

III. $\sum_{n=1}^{\infty} \frac{e^n}{n!}$

$\lim_{n\to\infty} \frac{e^n \cdot e^1}{(n+1)n!} \cdot \frac{n!}{e^n}$

$\lim_{n\to\infty} \frac{e^1}{n+1} = 0$

< 1

Converges

A. I only

B. II only

C. I and III only

D. I and II only

E. I, II, and III

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10. Apply any appropriate test to determine which of the following series diverges.

I. $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$

Limit Comparison to $\frac{1}{n}$ (which diverges)

$\lim_{n\to\infty} \frac{n}{2n^2+1} \cdot \frac{n}{1} = \frac{1}{2}$

Finite and positive

$\therefore$ both diverge

II. $\sum_{n=1}^{\infty} \frac{n!}{9^n}$

ratio test

$\lim_{n\to\infty} \frac{(n+1)n!}{9^n \cdot 9^1} \cdot \frac{9^n}{n!}$

$\lim_{n\to\infty} \frac{n+1}{9} = \infty > 1$

diverges

III. $\sum_{n=1}^{\infty} \frac{n+1}{4n+1}$

n^{th} term test

$\lim_{n\to\infty} a_n \neq 0$

$\therefore$ diverges

A. I only

B. II only

C. III only

D. I and II only

E. I, II, and III

Match the test for convergence of an infinite series with the conditions of convergence.

Convergence Test		Condition of convergence
11. F	nth-Term Test	~~A.~~ $p > 1$
12. E	Geometric Series	~~B.~~ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$
13. A	p-series	~~C.~~ $0 < a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ converges
14. G	Alternating Series Test	~~D.~~ $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges
15. H	Integral Test	~~E.~~ $\|r\| < 1$
16. B	Ratio Test	~~F.~~ Inconclusive for convergence
17. C	Comparison Test	G. $\|a_{n+1}\| \le \|a_n\|$ and $\lim_{n\to\infty} a_n = 0$
18. D	Limit Comparison Test	H. $\int_1^{\infty} f(x)\,dx$ converges.

10.8 Ratio Test

Test Prep

19. If the Ratio Test is applied to the series $\sum_{n=1}^{\infty} \frac{7^n}{(n+1)!}$, which of the following limits results, implying that the series converges?

$$a_n = \frac{7^{n+1}}{(n+2)!} \qquad \lim_{n\to\infty} \frac{\cancel{7^n}\cdot 7}{(n+2)\cancel{(n+1)}\cancel{n!}} \cdot \frac{\cancel{(n+1)}\cancel{n!}}{\cancel{7^n}}$$

$$\lim_{n\to\infty} \frac{7}{n+2}$$

A. $\lim_{n\to\infty} \frac{7^n}{(n+1)!}$ B. $\lim_{n\to\infty} \frac{7}{n+2}$ C. $\lim_{n\to\infty} \frac{(n+1)!}{7^n}$ D. $\lim_{n\to\infty} \frac{n+2}{7}$

20. Use the Ratio Test to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

$$a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!} = \frac{(n+1)^n \cdot \cancel{(n+1)}}{\cancel{(n+1)}\cdot n!} = \frac{(n+1)^n}{n!}$$

$$\lim_{n\to\infty} \frac{(n+1)^n}{n!} \cdot \frac{n!}{n^n}$$

$$\lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \underbrace{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n}_{\text{Definition of "e"}} = e > 1$$

$\boxed{\text{Diverges}}$