4.6 Approximating with Local Linearity

Calculus

Solutions

Practice

For each differential equation, let $y = f(x)$ be the particular solution to the differential equation with the given initial condition.

1. $\frac{dy}{dx} = (5-y)\sin x$ and $f\left(\frac{\pi}{2}\right) = 2$.

   a. Write an equation for the line tangent to the graph of f at the point $\left(\frac{\pi}{2}, 2\right)$.

   $m = (5-2)\sin\left(\frac{\pi}{2}\right) = 3$

   $y - 2 = 3\left(x - \frac{\pi}{2}\right)$

   or

   $\boxed{y = 3x - \frac{3\pi}{2} + 2}$

   b. Use the tangent line to approximate $f(1.5)$.

   $\boxed{1.7876}$

2. $\frac{dy}{dx} = -\frac{4x}{y}$ and $f(1) = 3$.

   a. Write an equation for the line tangent to the graph of f at the point $(1,3)$.

   $m = -4/3$

   $y - 3 = -4/3(x-1)$

   $\boxed{y = -\frac{4}{3}x + 13/3}$

   b. Use the tangent line to approximate $f(1.1)$.

   $\boxed{2.8667}$

Answer the questions for each function listed.

3. $f(x) = 2\cos x + 1$ is concave down on $\left[0, \frac{\pi}{2}\right]$.

   a. What is the estimate for $f(1)$ using the local linear approximation for f at $x = \frac{\pi}{2}$? Give an exact answer (no rounding).

   $f\left(\frac{\pi}{2}\right) = 2\cos\left(\frac{\pi}{2}\right) + 1 = 1 \leftarrow y_1$

   $f'\left(\frac{\pi}{2}\right) = -2\sin\left(\frac{\pi}{2}\right) = -2 \leftarrow m$

   $y - 1 = -2\left(x - \frac{\pi}{2}\right)$

   $y - 1 = -2\left(1 - \frac{\pi}{2}\right)$

   $y - 1 = -2 + \pi$

   $\boxed{y = \pi - 1}$

   b. Is it an underestimate or overestimate? Explain.

   Overestimate because $f(x)$ is concave down

4. $f(x) = \frac{e^{2x}}{x+1}$ is concave up on $x > -1$.

   a. What is the estimate for $f(0.1)$ using the local linear approximation for f at $x = 0$?

   $f(0) = \frac{e^0}{0+1} = 1 \leftarrow y_1$

   $f'(x) = \frac{2e^{2x}(x+1) - e^{2x}(1)}{(x+1)^2}$

   $f'(0) = \frac{2e^0(1) - e^0}{(1)^2} = 1 \leftarrow m$

   $y - 1 = 1(x - 0)$

   $y = x + 1$

   $\boxed{y = 1.1}$

   b. Is it an underestimate or overestimate? Explain.

   Underestimate because $f(x)$ is concave up.

5. $f(x) = -\sqrt{4-x}$ is concave up on its domain.
   a. What is the estimate for $f(1.9)$ using the local linear approximation for f at $x = 2$? Round to three decimal places.

$f(2) = -\sqrt{2} \leftarrow y_1$

$f'(x) = -\frac{1}{2\sqrt{4-x}}(-1) = \frac{1}{2\sqrt{4-x}}$

$f'(2) = \frac{1}{2\sqrt{2}} \leftarrow m$

$y + \sqrt{2} = \frac{1}{2\sqrt{2}}(x-2)$

$\boxed{y = -1.4495}$

   b. Is it an underestimate or overestimate? Explain.

Underestimate. $f(x)$ is concave up.

6. f is concave down and $f(3) = -1$ and $f'(3) = 2$.
   a. What is the estimate for $f(3.2)$ using the local linear approximation for f at $x = 3$?

$y+1 = 2(x-3)$

$y = 2x - 7$

$y = 2(3.2) - 7$

$\boxed{y = -0.6}$

   b. Is it an underestimate or overestimate? Explain.

Overestimate. $f(x)$ is concave down.

7. f is concave up and $f(-5) = 2$ and $f'(-5) = -1$.
   a. What is the estimate for $f(-5.1)$ using the local linear approximation for f at $x = -5$?

$y - 2 = -1(x+5)$

$y = -x - 3$

$y = 5.1 - 3$

$\boxed{y = 2.1}$

   b. Is it an underestimate or overestimate? Explain.

Underestimate. $f(x)$ is concave up.

8. f is concave down and $f(2) = 1$ and $f'(2) = -3$.
   a. What is the estimate for $f(1.9)$ using the local linear approximation for f at $x = 2$?

$y - 1 = -3(x-2)$

$y = -3x + 7$

$y = -3(1.9) + 7$

$\boxed{y = 1.3}$

   b. Is it an underestimate or overestimate? Explain.

Overestimate. $f(x)$ is concave down.

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$f(1) = 3 - 4 + 2 = 1$

Test Prep

9. Let f be the function given by $f(x) = 3x^2 - 4x + 2$. The tangent line to the graph of f at $x = 1$ is used to approximate values of $f(x)$. Which of the following is the smallest value of x for which the error resulting from this tangent line approximation is more than 0.5?
[*Hint for your calculator use*: Create a table to compare values of two functions.]

$f'(x) = 6x - 4$

$f'(1) = 2$

$y - 1 = 2(x-1)$

$y = 2x - 1$

(A) 1.3 (B) 1.4 (C) 1.5 (D) 1.6 (E) 1.7

10. The depth of snow in a field is given by the twice-differentiable function S for $0 \le t \le 12$, where $S(t)$ is measured in centimeters and time t is measured in hours. Values of $S'(t)$, the derivative of S, at selected values of time t are shown in the table above. It is known that the graph of S is concave down for $0 \le t \le 12$.

t (hours)	0	1	4	9	12
$S'(t)$ (centimeters per hour)	1.8	2.4	2.0	1.6	1.3

a. Use the data in the table to approximate $S''(10)$. Show the computations that lead to your answer. Using correct units, explain the meaning of $S''(10)$ in the context of the problem.

$$S''(10) \approx \frac{S'(12) - S'(9)}{12 - 9} = \frac{-0.3}{3} = \boxed{-0.1 \text{ cm/hr}^2}$$

The rate of the depth of snow is decreasing by 0.1 cm per hour per hour.

b. Is there a time t, for $0 \le t \le 12$, at which the depth of snow is changing at a rate of 1.5 centimeters per hour? Justify your answer?

Yes, on the interval $9 \le t \le 12$. By the Intermediate Value Theorem the rate must be 1.5.

c. At time $t = 4$, the depth of snow is 28 centimeters. Use the line tangent to the graph of S at $t = 4$ to approximate the depth of the snow at time $t = 6$. Is the approximation an underestimate or an overestimate of the actual depth of snow at time $t = 6$? Justify your answer.

$$S - 28 = 2(t - 4)$$
$$S = 2t + 20$$
$$\boxed{S = 32 \text{ cm}}$$

32 cm is an overestimate because $S(t)$ is concave down