5.11 Solving Optimization Problems

Calculus

Solutions

Practice

1. A particle is traveling along the x-axis and it's position from the origin can be modeled by $x(t) = -\frac{2}{3}t^3 + t^2 + 12t + 1$ where x is meters and t is minutes on the interval .
   a. At what time t during the interval $0 \le t \le 4$ is the particle farthest to the left? min.

$V(t) = -2t^2 + 2t + 12$

$-2(t^2 - t - 6) = 0$

$-2(t-3)(t+2) = 0$

$t = 3 \quad t = -2$

$x(0) = 1$

$x(3) = 28$

$x(4) = 22.333$

$\boxed{t = 0}$

   b. On the same interval what is the particle's maximum speed?

$a(t) = -4t + 2 = 0$

$-4t = -2$

$t = \frac{1}{2}$

$V(0) = 12$

$V(\frac{1}{2}) = 12.5$

$V(4) = -12$

$\boxed{12.5 \text{ meters/min}}$

2. Find the point on the graph of the function $f(x) = x^2$ that is closest to the point $\left(2, \frac{1}{2}\right)$.

$D = \sqrt{(x-2)^2 + (x^2 - \frac{1}{2})^2}$

$\frac{dD}{dx} = \frac{2(x-2) + 2(x^2 - \frac{1}{2})(2x)}{2\sqrt{(x-2)^2 + (x^2 - \frac{1}{2})^2}}$

$\frac{4x^3 - 4}{2\sqrt{(x-2)^2 + (x^2 - \frac{1}{2})^2}} = 0$

$4x^3 - 4 = 0$

$x = 1$

$\boxed{(1, 1)}$

3. A particle moves along the x-axis so that at any time t its position is $s(t) = \frac{1}{3}t^3 - 4t^2 + 7t - 5$ where s is inches and t is hours.
   a. At what time t during the interval $0 \le t \le 6$ is the particle farthest to the right?

$V(t) = t^2 - 8t + 7$

$(t-7)(t-1) = 0$

$t = 7 \quad t = 1$

$s(0) = -5$

$s(1) = -1.667$

$s(6) = -35$

$\boxed{t = 1}$

   b. On the same interval what is the particle's maximum speed?

$a(t) = 2t - 8 = 0$

$t = 4$

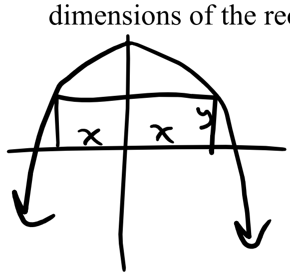
$v(0) = 7$

$v(4) = -9$

$v(6) = -5$

$\boxed{9 \text{ inches/hour}}$

4. A rectangle is formed with the base on the x-axis and the top corners on the function $y = 20 - x^2$. Find the dimensions of the rectangle with the largest area.



$A = (x + x) \cdot y$

$A = 2x(20 - x^2)$

$A = 40x - 2x^3$

$A' = 40 - 6x^2 = 0$

$-6x^2 = -40$

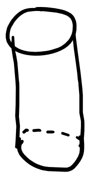
$x^2 = 20/3$

$x = \pm\sqrt{20/3}$

Width: $\boxed{y = \frac{40}{3}}$

Length: $2x = \boxed{2\sqrt{\frac{20}{3}}}$

5. What is the radius of a cylindrical soda can with volume of 512 cubic inches that will use the minimum material? Volume of a cylinder is $V = \pi r^2 h$. Surface area of a cylinder is $A = 2\pi r^2 + 2\pi rh$

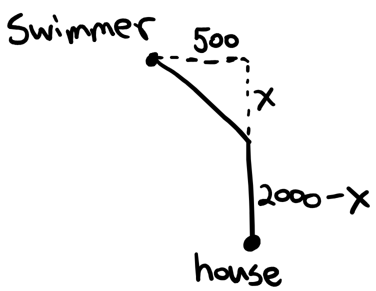


$$512 = \pi r^2 h$$
$$\frac{512}{\pi r^2} = h$$

$$A = 2\pi r^2 + 2\pi r\left(\frac{512}{\pi r^2}\right)$$
$$A = 2\pi r^2 + \frac{1024}{r}$$
$$A' = 4\pi r - \frac{1024}{r^2} = 0$$
$$4\pi r = \frac{1024}{r^2}$$
$$r^3 = \frac{256}{\pi}$$

$$\boxed{r = \sqrt[3]{\frac{256}{\pi}} \approx 4.335 \text{ inches}}$$

6. A swimmer is 500 meters from the closest point on a straight shoreline. She needs to reach her house located 2000 meters down shore from the closest point. If she swims at $\frac{1}{2}$ m/s and she runs at 4 m/s, how far from her house should she come ashore so as to arrive at her house in the shortest time? *Hint*: time $= \frac{\text{distance}}{\text{rate}}$



$$T = \frac{\sqrt{500^2 + x^2}}{\frac{1}{2}} + \frac{2000 - x}{4}$$
$$T' = \frac{2x}{\sqrt{500^2 + x^2}} - \frac{1}{4} = 0$$

Use a Calculator!

$$x = 62.994$$

$$\boxed{1937.006 \text{ meters from her house.}}$$

7. Mr. Kelly is selling licorice for \$1.50 per piece. The cost of producing each piece of licorice increases the more he produces. Mr. Kelly finds that the total cost to produce the licorice is $10\sqrt{x}$ dollars, where x is the number of licorice pieces. What is the most Mr. Kelly could lose per piece on the sale of licorice. Justify your answer. (hint: profit is the difference between money received and the cost of the licorice.)

$$P = 1.50x - 10\sqrt{x}$$
$$P' = 1.5 - (10)\frac{1}{2\sqrt{x}} = 0$$
$$\frac{5}{\sqrt{x}} = 1.5$$
$$\frac{10}{3} = \sqrt{x}$$
$$x = \frac{100}{9} \approx 11.111$$

$$\boxed{P(11.111) = -\$16.67}$$

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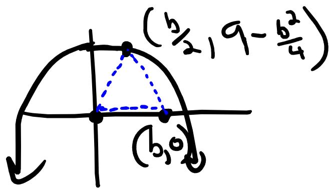
Test Prep

8. Let $f(x) = xe^{-x} + ce^{-x}$, where c is a positive constant. For what positive value of c does f have an absolute maximum at $x = -5$?

$$f'(x) = (1)e^{-x} + xe^{-x}(-1) + ce^{-x}(-1)$$
$$f'(x) = e^{-x}(1 - c - x)$$
$$0 = e^{5}(1 - c + 5)$$
$$0 = 6 - c$$
$$\boxed{6 = c}$$

9. Let $f(x) = 9 - x^2$ for $x \geq 0$ and $f(x) \geq 0$. An isosceles triangle whose base is the interval from the point $(0, 0)$ to the point $(b, 0)$ has its vertex on the graph of f. For what value of b does the triangle have maximum area? Recall that the area of a triangle is modeled by $A = \frac{1}{2}(\text{base})(\text{height})$.



$$A = \frac{1}{2}(b)\left(9 - \frac{b^2}{4}\right)$$
$$A = \frac{9}{2}b - \frac{1}{8}b^3$$
$$A' = \frac{9}{2} - \frac{3}{8}b^2$$

$$\frac{9}{2} - \frac{3}{8}b^2 = 0$$
$$b^2 = \frac{9}{2} \cdot \frac{8}{3}$$
$$b^2 = 12$$
$$\boxed{b = \sqrt{12}}$$

10. Mr. Sullivan is making apple juice from the apples he collected in his neighbor's orchard. The number of gallons of apple juice in a tank at time t is given by the twice-differentiable function A, where t is measured in days and $0 \leq t \leq 20$. Values of $A(t)$ at selected times t are given in the table below.

t (days)	0	3	8	12	20
$A(t)$ (gallons)	2	6	9	10	7

a. Use the data in the table to estimate the rate at which the number of gallons of apple juice in the tank is changing at time $t = 10$ days. Show the computations that lead to your answer. Indicate units of measure.

$$\frac{A(12) - A(8)}{12 - 8} = \frac{10 - 9}{4} = \frac{1}{4} \text{ gallon per day}$$

b. For $0 \leq t \leq 12$, is there a time t at which $A'(t) = \frac{2}{3}$? Justify your answer.

$$\frac{10-2}{12-0} = \frac{8}{12} = \frac{2}{3}$$

Yes, because of the Mean Value Theorem on the interval $[0, 12]$.

c. The number of gallons of apple juice in the tank at time t is also modeled by the function B defined by $B(t) = 3t - \frac{1}{2}(t+4)^{\frac{3}{2}} + 6$, where t is measured in days and $0 \leq t \leq 20$. Based on the model, at what time t, for $0 \leq t \leq 20$, is the number of gallons of apple juice in the tank an absolute maximum?

$$B'(t) = 3 - \frac{3}{4}(t+4)^{\frac{1}{2}} = 0$$
$$\sqrt{t+4} = 3 \cdot \frac{4}{3}$$
$$t + 4 = 16$$
$$t = 12$$

$B(0) = 2$

$B(12) = 10 \leftarrow$ max

$B(20) \approx 7.212$

$\boxed{\text{At } t = 12}$

d. For the function B defined in part c, the locally linear approximation near $t = 5$ is used to approximate $B(5)$. Is this approximation an overestimate or an underestimate for the value of $B(5)$? Give a reason for your answer.

$$B''(5) = -\frac{3}{8}\frac{1}{\sqrt{9}} = -\frac{1}{8}$$

$\boxed{\text{Overestimate because } B'' < 0, \text{ which means } B(t) \text{ is concave down.}}$