5.12 Behaviors of Implicit Relations
Calculus

Solutions

Practice

Consider the curves in the xy-plane for each problem. At the point given point, is the curve increasing or decreasing? Justify your answer.

1. $x^2 - \frac{y^2}{2} = -1$ at $(-1, 2)$

$$2x - y\frac{dy}{dx} = 0$$

$$2(-1) - (2)\frac{dy}{dx} = 0$$

$$-2\frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = -1$$

$\boxed{\text{Decreasing because } \left.\frac{dy}{dx}\right|_{(-1,2)} < 0}$

2. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 5$ at $(1, -8)$

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0$$

$$\frac{2}{3\sqrt[3]{1}} + \frac{2}{3\sqrt[3]{-8}}\frac{dy}{dx} = 0$$

$$\frac{2}{-6}\frac{dy}{dx} = -\frac{2}{3}$$

$$\frac{dy}{dx} = 2$$

$\boxed{\text{Increasing because } \left.\frac{dy}{dx}\right|_{(1,-8)} > 0}$

3. $x^2 - 2xy + y^2 = 1$ at $(-1, -2)$

$$2x - 2\left[(1)y + x\frac{dy}{dx}\right] + 2y\frac{dy}{dx} = 0$$

$$2(-1) - 2\left[(-2) + (-1)\frac{dy}{dx}\right] + 2(-2)\frac{dy}{dx} = 0$$

$$-2 + 4 + 2\frac{dy}{dx} - 4\frac{dy}{dx} = 0$$

$$-2\frac{dy}{dx} = -2$$

$$\frac{dy}{dx} = 1$$

$\boxed{\text{Increasing because } \left.\frac{dy}{dx}\right|_{(-1,-2)} > 0.}$

Consider the given differential equation $\frac{dy}{dx}$, where $y = f(x)$ is a particular solution with a given point. For each problem, determine if f has a relative minimum, a relative maximum, or neither at the given point. Justify your answer.

4. $\frac{dy}{dx} = y \sin x$ where $f(2\pi) = 1$

$$1 \cdot \sin(2\pi) = 0$$

$$\frac{dy}{dx} = 0 \checkmark$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}\sin x + y\cos x$$

$$(y\sin x)(\sin x) + y\cos x$$

$$1 \cdot \sin^2(2\pi) + 1 \cdot \cos(2\pi)$$

$$0 + 1$$

$$1$$

$\boxed{\text{Relative minimum b/c } \frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} > 0.}$

Instructions continued from last page.

5. $\frac{dy}{dx} = \frac{x}{y} + \ln x$ where $f(1) = -2$

$\frac{1}{-2} + \ln(1)$

$-\frac{1}{2} + 0$

$\boxed{\text{Neither. } \frac{dy}{dx} \neq 0.}$

6. $\frac{dy}{dx} = yx^2$ where $f(0) = -5$

$(-5)(0) = 0$

$\frac{dy}{dx} = 0$ ✓

$\frac{d^2y}{dx^2} = \frac{dy}{dx} x^2 + y(2x)$

$(yx^2)x^2 + 2xy$

$0 + 0$

$\boxed{\text{Neither. } \frac{d^2y}{dx^2} = 0 \text{ therefore } f \text{ is not concave up or down.}}$

5.12 Behaviors of Implicit Relations

Test Prep

7. Consider the curve defined by $x^2 - y^2 - 5xy = 25$.

a. Show that $\frac{dy}{dx} = \frac{2x-5y}{5x+2y}$

$2x - 2y\frac{dy}{dx} - 5y - 5x\frac{dy}{dx} = 0$

$-2y\frac{dy}{dx} - 5x\frac{dy}{dx} = 5y - 2x$

$\frac{dy}{dx}(-2y - 5x) = 5y - 2x$

$\frac{dy}{dx} = \frac{5y-2x}{-2y-5x} = \frac{2x-5y}{5x+2y}$

multiply by $\frac{-1}{-1}$

b. Find the slope of the line tangent to the curve at each point on the curve when $x = 2$.

$(2)^2 - y^2 - 5(2)y = 25$

$4 - y^2 - 10y = 25$

$-y^2 - 10y - 21 = 0$

$-(y^2 + 10y + 21) = 0$

$-(y+3)(y+7) = 0$

$y = -3 \quad y = -7$

$\frac{dy}{dx}_{(2,-3)} = \frac{2(2) - 5(-3)}{5(2) + 2(-3)}$

$\frac{4+15}{10-6} = \boxed{\frac{19}{4}}$

$\frac{dy}{dx}_{(2,-7)} = \frac{2(2) - 5(-7)}{5(2) + 2(-7)} = \boxed{\frac{39}{-4}}$

c. Find the positive value of x at which the curve has a vertical tangent line. Show the work that leads to your answer.

Vertical tangent if $\frac{dy}{dx}$ has a denominator $= 0$,

or $2y + 5x = 0$.

$$2y = -5x$$

$$y = -\frac{5x}{2}$$

Substitute "y" into original equation

$$x^2 - \left(-\frac{5x}{2}\right)^2 - 5x\left(-\frac{5x}{2}\right) = 25$$

$$x^2 - \frac{25}{4}x^2 + \frac{25}{2}x^2 = 25$$

$$\frac{4}{4}x^2 - \frac{25}{4}x^2 + \frac{50}{4}x^2 = 25$$

$$\frac{29}{4}x^2 = 25$$

$$x^2 = \frac{100}{29}$$

$$\boxed{x = \sqrt{\frac{100}{29}}}$$

d. Let x and y be functions of time t that are related by the equation $x^2 - y^2 - 5xy = 25$. At time $t = 3$, the value of x is 5, the value of y is 0, and the value of $\frac{dy}{dt}$ is -2. Find the value of $\frac{dx}{dt}$ at time $t = 3$.

$$2x\frac{dx}{dt} - 2y\frac{dy}{dt} - 5\left[\frac{dx}{dt}y + x\frac{dy}{dt}\right] = 0$$

$$2(5)\frac{dx}{dt} - 2(0)(-2) - 5\left[\frac{dx}{dt}(0) + (5)(-2)\right] = 0$$

$$10\frac{dx}{dt} - 5(-10) = 0$$

$$10\frac{dx}{dt} = -50$$

$$\boxed{\frac{dx}{dt} = -5}$$