Calculus

# 5.1 The Mean Value Theorem

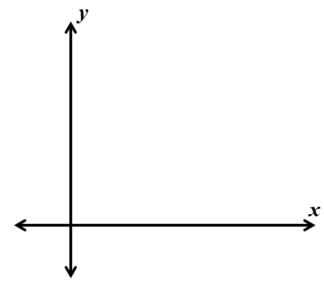
**Notes**

Write your questions and thoughts here!

We use the MVT to justify conclusions about a function over an interval.

**Mean Value Theorem:**

If a function $f$ is continuous over the interval and differentiable over the interval then there exists a point within that open interval where the instantaneous rate of change equals the average rate of change over the interval.




1. Use the function $f(x) = -x^2 + 3x + 10$ to answer the following.
   a. On the interval $[2, 6]$, what is the average rate of change?

   b. On the interval $(2, 6)$, when does the instantaneous rate of change equal the average rate of change?

## MVT vs IVT

| **Mean Value Theorem**<br>MVT | **Intermediate Value Theorem**<br>IVT |
|---|---|
| • The derivative (instantaneous rate of change) must equal the average rate of change somewhere in the interval. | • On a given interval, you will have a $y$-value at each of the end points of the interval. Every $y$-value exists between these two $y$-values at least once in the interval. |

2.

| $t$ minutes | 0 | 5 | 15 | 20 | 30 |
|---|---|---|---|---|---|
| $h(t)$ feet | 0 | 40 | 70 | 65 | 80 |

A hot air balloon is launched into the air with a human pilot. The twice-differentiable function $h$ models the balloon's height, measured in feet, at time $t$, measured in minutes. The table above gives values of the $h(t)$ of the balloon at selected times $t$.

a. For $5 \le t \le 15$, must there be a time $t$ when the balloon is $50$ feet in the air? Justify your answer.

b. For $20 \le t \le 30$, must there be a time $t$ when the balloon's velocity is $1.5$ feet per minute? Justify your answer.

## 5.1 The Mean Value Theorem

**Practice**

Calculus

1. Skater Sully is riding a skateboard back and forth on a street that runs north/south. The twice-differentiable function $S$ models Sully's position on the street, measured by how many meters north he is from his starting point, at time $t$, measured in seconds from the start of his ride. The table below gives values of $S(t)$ at selected times $t$.

| $t$ seconds | 0 | 20 | 30 | 60 |
|---|---|---|---|---|
| $S(t)$ meters | 0 | $-5$ | 7 | 40 |

a. For $0 \le t \le 20$, must there be a time $t$ when Sully is 2 meters south of his starting point? Justify your answer.

b. For $30 \le t \le 60$, must there be a time $t$ when Sully's velocity is 1.1 meters per second? Justify your answer.

2. A particle is moving along the $x$-axis. The twice-differentiable function $s$ models the particles distance from the origin, measured in centimeters, at time $t$, measured in seconds.

| $t$ seconds | 3 | 10 | 20 | 25 |
|---|---|---|---|---|
| $s(t)$ cm | 5 | $-2$ | $-10$ | 8 |

a. For $20 \leq t \leq 25$, must there be a time $t$ when the particle is at the origin? Justify your answer.

b. For $3 \leq t \leq 10$, must there be a time $t$ when the particle's velocity is $-1$ cm per second? Justify your answer.

3. A hot air balloon is launched into the air with a human pilot. The twice-differentiable function $h$ models the balloon's height, measured in feet, at time $t$, measured in minutes. The table below gives values of $h(t)$ at selected times $t$.

| $t$ minutes | 0 | 6 | 10 | 40 |
|---|---|---|---|---|
| $h(t)$ feet | 0 | 46 | 35 | 125 |

a. For $6 \leq t \leq 10$, must there be a time $t$ when the balloon is 50 feet in the air? Justify your answer.

b. For $10 \leq t \leq 40$, must there be a time $t$ when the balloon's velocity is 3 feet per second? Justify your answer.

**Using the Mean Value Theorem, find where the instantaneous rate of change is equivalent to the average rate of change.**

4. $y = x^2 - 5x + 2$ on $[-4, -2]$

5. $y = \sin 3x$ on $[0, \pi]$

6. $y = (-5x + 15)^{\frac{1}{2}}$ on $[1, 3]$

7. $y = e^x$ on $[0, \ln 2]$

## 5.1 The Mean Value Theorem

**Test Prep**

Calculator active problem

8. A particle moves along the $x$-axis so that its position at any time $t \geq 0$ is given by $x(t) = t^3 - 3t^2 + t + 1$. For what values of $t$, $0 \leq t \leq 2$, is the particle's instantaneous velocity the same as its average velocity on the closed interval $[0, 2]$?

No calculator on this problem.

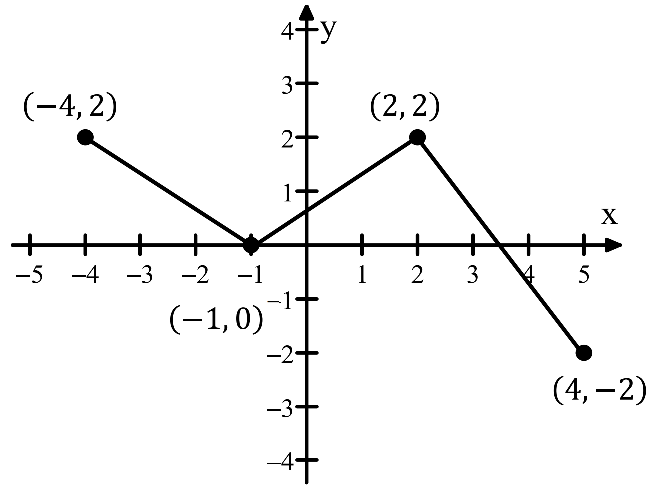
9. The table below gives selected values of a function $f$. The function is twice differentiable with $f''(x) > 0$.

| $x$ | $f(x)$ |
|---|---|
| 3 | 12.5 |
| 5 | 13.9 |
| 7 | 16.1 |

Which of the following could be the value of $f'(5)$?

(A) 0.5 (B) 0.7 (C) 0.9 (D) 1.1 (E) 1.3

10. Let $g$ be a continuous function. The graph of the piecewise-linear function $g'$, the derivative of $g$, is shown above for $-4 \leq x \leq 4$. Find the average rate of change of $g'(x)$ on the interval $-4 \leq x \leq 4$. Does the Mean Value Theorem applied on the interval $-4 \leq x \leq 4$ guarantee a value of $c$, for $-4 < x < 4$, such that $g''(c)$ is equal to this average rate of change? Why or why not?




Graph of $g'$

---

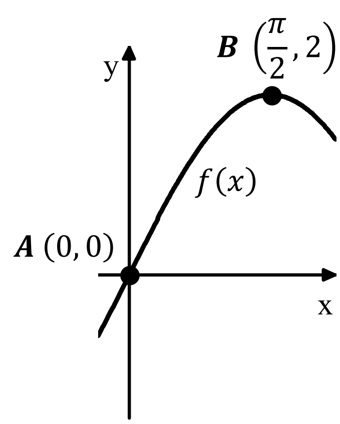
11.

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 1 | 3 | 8 | 2 | 4 |
| 2 | 6 | 3 | 1 | 2 |
| 3 | 5 | $-3$ | 6 | 3 |
| 4 | $-2$ | 6 | 3 | 5 |

The functions $f$ and $g$ are differentiable for all real numbers. The table above gives values of the functions and their first derivatives at selected values of $x$. The function $h$ is given by $h(x) = f\big(g(x)\big) + 2$. Must there be a value $c$ for $2 < c < 4$ such that $h'(c) = 1$.

---

12. **Calculator active problem.** Let $f$ be the function given by $f(x) = 2\sin x$. As shown above, the graph $f$ crosses the origin at point $A$ and point $B$ at the coordinate point $\left(\frac{\pi}{2}, 2\right)$. Find the $x$-coordinate of the point on the graph of $f$, between points $A$ and $B$, at which the line tangent to the graph of $f$ is parallel to line $AB$. Round or truncate to three decimals.



13. A differentiable function $g$ has the property that $g'(x) > 2$ for $1 \leq x \leq 5$ and $g(4) = 3$. Which of the following could be true?

I. $g(1) = -6$

II. $g(2) = 0$

III. $g(5) = 4$

(A) I only

(B) II only

(C) I and II only

(D) I and III only

(E) II and III only

---

14. **Calculator active problem.** Let $f$ be the function with $f(1) = e$, $f(4) = \frac{1}{e}$, and derivative given by $f'(x) = (x-1)\sin(ex)$. How many values of $x$ in the open interval $(1, 4)$ satisfy the conclusion of the Mean Value Theorem for the function $f$ on the closed interval $[1, 4]$?

(A) None

(B) One

(C) Two

(D) More than two