5.4 The First Derivative Test

Calculus

Practice

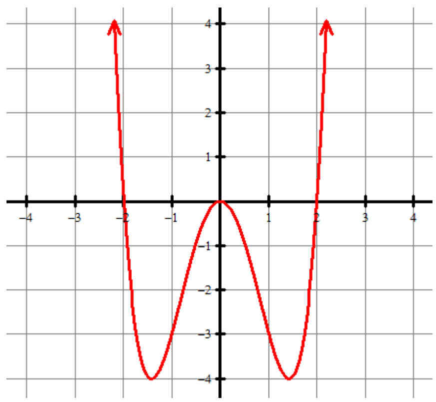
1. Assume $f(x)$ is continuous for all real numbers. The sign of its derivative is given in the table below for the domain of f. Identify all relative extrema and justify your answers.

Interval	$(-\infty, -2)$	$(-2, 0)$	$(0, 3)$	$(3, \infty)$
$f'(x)$	**Positive**	**Negative**	**Negative**	**Positive**

max at $x=-2$ b/c f' changes sign from pos to neg.
min at $x=3$ b/c f' changes sign from neg to pos.

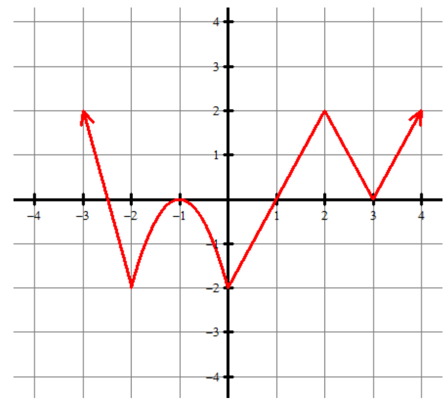
For each problem, the graph of f', the derivative of f, is shown. Find all relative max/min of f and justify.

2.



max at $x=-2$. f' changes from pos to neg.
Min at $x=2$. f' changes from neg. to pos.

3.



max at $x=-2.5$ because f' changes sign from pos to neg
min at $x=1$ because f' changes sign from neg to pos.

For each problem, the derivative of a function g is given. Find all relative max/min of g and justify.

4. $g'(x) = (x+4)e^x = 0$

$x+4=0$ $\quad e^x=0$ ← not possible
$x=-4$

x	$(-\infty,-4)$	-4	$(-4,\infty)$
f'	neg.	0	pos.

minimum at $x=-4$ because f' changes sign from negative to positive.

5. $g'(x) = x^2 + 5x + 4$

$(x+4)(x+1)=0$
$x=-4 \quad x=-1$

x	$(-\infty,-4)$	-4	$(-4,-1)$	-1	$(-1,\infty)$
f'	pos	0	neg	0	pos

max at $x=-4$ b/c f' changes sign from pos. to neg.
Min at $x=-1$ b/c f' changes sign from neg to pos

Use a calculator to help find all x-values of relative max/min of f. No justification necessary.

6. $f'(x) = x^3 - 6\cos(x^2) + 2$

min at $x=-1.922$ and $x=1.018$
max at $x=-1.250$

7. $f'(x) = \frac{2-\ln x}{x^2}$

max at $x=7.389$

8. $f'(x) = \sqrt{x^4+2} + x^2 - 5x$

min at $x=2.467$
max at $x=0.3016$

Use the First Derivative Test to locate the x-value of all extrema. Classify if it is a relative max or min and justify your answer.

9. $f(x) = x^3 - 12x + 1$

$f'(x) = 3x^2 - 12$

$3x^2 - 12 = 0$

$x = \pm 2$

x	$(-\infty,-2)$	-2	$(-2,2)$	2	$(2,\infty)$
f'	pos	0	neg	0	pos

max at $x=-2$ b/c f' changes sign from pos. to neg.

min at $x=2$ b/c f' changes sign from neg. to pos.

10. $g(x) = xe^{5x}$

$g'(x) = e^{5x} + xe^{5x}(5)$

$e^{5x}(1+5x) = 0$

$e^{5x} = 0$ $\quad$ $1+5x=0$

not possible

$x = -\frac{1}{5}$

x	$(-\infty,-\frac{1}{5})$	$-\frac{1}{5}$	$(-\frac{1}{5},\infty)$
g'	neg	0	pos

min at $x=-\frac{1}{5}$ because g' changes sign from negative to positive

11. $h(x) = \frac{x^3}{x+1}$

$h'(x) = \frac{3x^2(x+1) - x^3(1)}{(x+1)^2} = \frac{3x^3+3x^2-x^3}{(x+1)^2}$

$h'(x) = \frac{2x^3+3x^2}{(x+1)^2} = 0$ $\quad$ $x^2(2x+3)=0$

$x=0$ $\quad$ $x=-\frac{3}{2}$ $\quad$ not a C.P. $x=-1$ not in domain of $h(x)$.

x	$(-\infty,-\frac{3}{2})$	$-\frac{3}{2}$	$(-\frac{3}{2},-1)$	-1	$(-1,0)$	0	$(0,\infty)$
h'	neg	0	pos	und.	pos	0	pos

min at $x=-\frac{3}{2}$ b/c $h'(x)$ changes sign from neg. to pos.

12. $f(x) = (x-5)^{\frac{2}{3}}$

$f'(x) = \frac{2}{3}(x-5)^{-\frac{1}{3}}$

f' does not exist at $x=5$

x	$(-\infty,5)$	5	$(5,\infty)$
f'	neg	0	pos

min at $x=5$ b/c f' changes sign from neg to pos.

13. What is the maximum value of $g(x) = 2\cos x$ on the open interval $(-\pi,\pi)$?

$g'(x) = -2\sin x = 0$

$x = -\pi,\ 0,$ and π

x	$(-\pi,0)$	0	$(0,\pi)$
$g'(x)$	pos	0	neg

max

$g(0) = 2\cos(0) = \boxed{2}$

14. What is the relative minimum value of $h(x) = -x^3 + 6x^2 - 3$?

$h'(x) = -3x^2 + 12x = 0$

$-3x(x-4) = 0$

$x=0$ $\quad$ $x=4$

x	$(-\infty,0)$	0	$(0,4)$	4	$(4,\infty)$
$h'(x)$	neg	0	pos	0	neg

min

$h(0) = \boxed{-3}$

g(x) is always negative!

15. If g is a differentiable function such that $g(x) < 0$ for all real numbers x and if $f'(x) = (x^2 - x - 12)g(x)$, which of the following is true?

x	$(-\infty,-3)$	-3	$(-3,4)$	4	$(4,\infty)$
f'	neg	0	pos	0	neg

$(x-4)(x+3)$

$x=4 \quad x=-3$

(A) f has a relative maximum at $x = -3$ and a relative minimum at $x = 4$.

(B) f has a relative minimum at $x = -3$ and a relative maximum at $x = 4$.

(C) f has a relative maximum at $x = 3$ and a relative minimum at $x = -4$.

(D) f has a relative minimum at $x = 3$ and a relative maximum at $x = -4$.

(E) It cannot be determined if f has any relative extrema.

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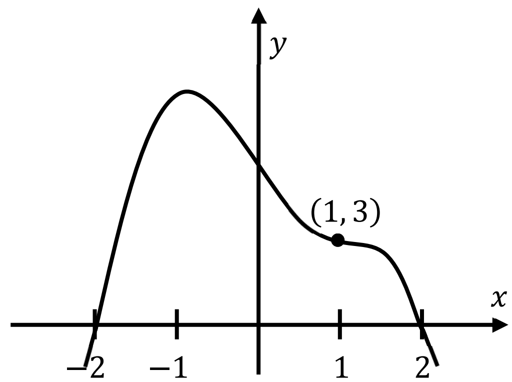
16. Let f be a twice-differentiable function defined on the interval $-2.1 < x < 2.1$ with $f(1) = -2$. The graph of f', the derivative of f, is shown above. The graph of f' crosses the x-axis at $x = -2$ and $x = 2$ and has a horizontal tangent at $x = -1$. Let g be the function given by $g(x) = e^{f(x)}$.



Graph of f'

(a) Write an equation for the line tangent to the graph of g at $x = 1$.

$g(1) = e^{-2}$

$g'(x) = e^{f(x)} \cdot f'(x)$

$g'(1) = e^{-2} \cdot 3 = \frac{3}{e^2}$

$y - \frac{1}{e^2} = \frac{3}{e^2}(x-1)$

(b) Find the average rate of change of g', the derivative of g, over the interval $[-2,2]$.

$$\frac{g'(2) - g'(-2)}{2-(-2)} = \frac{e^{f(2)} \cdot (0) - e^{f(-2)} \cdot (0)}{4} = \boxed{0}$$

(c) For $-2.1 < x < 2.1$, find all values of x at which g has a local minimum. Justify your answer.

$g'(x) = e^{f(x)} f'(x)$

Always positive

$e^{f(x)} f'(x) = 0$

$x = -2$ and $x = 2$

x	$(-2.1,-2)$	-2	$(-2,2)$	2	$(2,2.1)$
$g'(x)$	neg	0	pos	0	neg

$\boxed{x=-2}$ b/c g' changes sign from neg to pos.

(d) The second derivative of g is $g''(x) = e^{f(x)}\left[\left(f'(x)\right)^2 + f''(x)\right]$. Is $g''(-1)$ positive, negative or zero? Justify your answer.

$e^{f(-1)}\left[f'(-1)^2 + f''(-1)\right]$

pos

pos b/c squared

zero because slope of f' is zero at $x=-1$

Positive