6.13 Improper Integrals
Calculus

Solutions

Practice

Evaluate each integral.

1. $\int_1^\infty \frac{1}{x^2}dx$

$\lim_{t\to\infty} \int_1^t x^{-2}\,dx$

$\lim_{t\to\infty} \left[-\frac{1}{x}\right]\Big|_1^t$

$\lim_{t\to\infty} \left[-\frac{1}{t} - -\frac{1}{1}\right]$

$0 + 1$

$\boxed{1}$

2. $\int_0^\infty \frac{2}{x^2+4x+3}dx$

Partial Fractions!

$\left[\frac{2}{x^2+4x+3} = \frac{A}{x+3} + \frac{B}{x+1}\right](x+3)(x+1)$

$2 = A(x+1) + B(x+3)$

Let $x=-3$ — Let $x=-1$

$2 = A(-2)$ — $2 = B(2)$

$-1 = A$ — $1 = B$

$\lim_{t\to\infty} \int_0^t -\frac{1}{x+3} + \frac{1}{x+1}\,dx$

$\lim_{t\to\infty} \left[-\ln|x+3| + \ln|x+1|\right]\Big|_0^t$

$\lim_{t\to\infty} \left[-\ln|t+3| + \ln|t+1|\right] - \left[-\ln|3| + \ln|1|\right]$

$\lim_{t\to\infty} \left[\ln\left|\frac{t+1}{t+3}\right| + \ln 3\right]$

$\ln 1 + \ln 3$

$\boxed{\ln 3}$

3. $\int_0^1 \frac{x+1}{\sqrt{x^2+2x}}dx$

$x(x+2)$ → Discontinuity at $x=0$

u-Substitution
$u = x^2+2x$
$\frac{du}{2(x+1)} = dx$

$\lim_{t\to 0^+} \int_t^1 \frac{x+1}{\sqrt{u}} \frac{du}{2(x+1)}$

$\lim_{t\to 0^+} \int_{t^2+2t}^3 \frac{1}{2} u^{-\frac{1}{2}}\,du$

$\lim_{t\to 0^+} \frac{1}{2} \frac{u^{\frac{1}{2}}}{\frac{1}{2}} \Big|_{t^2+2t}^3$

$\lim_{t\to 0^+} \left[\sqrt{3} - \sqrt{t^2+2t}\right]$

$\boxed{\sqrt{3}}$

4. $\int_1^\infty xe^{-x}\,dx$

Integration by Parts!

$f = x$ — $g' = e^{-x}$

$f' = dx$ — $g = -e^{-x}$

$\lim_{t\to\infty} \left[-xe^{-x}\Big|_1^t - \int_1^t -e^{-x}dx\right]$

$\lim_{t\to\infty} \left[-xe^{-x} + (-e^{-x})\right]\Big|_1^t$

$\lim_{t\to\infty} \left[-te^{-t} - e^{-t}\right] - \left[-1e^{-1} - e^{-1}\right]$

$\lim_{t\to\infty} \left[-\frac{t}{e^t} - \frac{1}{e^t}\right] + \left[\frac{1}{e} + \frac{1}{e}\right]$

$0 - 0 + \frac{2}{e}$

$\boxed{\frac{2}{e}}$

5. $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$

$$\int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{\infty} \frac{1}{1+x^2} dx$$

$$\lim_{t \to -\infty} \int_t^0 \frac{1}{1+x^2} + \lim_{t \to \infty} \int_0^t \frac{1}{1+x^2} dx$$

$$\lim_{t \to -\infty} \tan^{-1}(x) \Big|_t^0 + \lim_{t \to \infty} \tan^{-1}(x) \Big|_0^t$$

$$\lim_{t \to -\infty} \left[\tan^{-1}(0) - \tan^{-1}(t)\right] + \lim_{t \to \infty} \left[\tan^{-1}(t) - \tan^{-1}(0)\right]$$

$$0 - \left(-\frac{\pi}{2}\right) + \left(\frac{\pi}{2}\right) - 0$$

$$\frac{\pi}{2} + \frac{\pi}{2}$$

$$\boxed{\pi}$$

6. $\int_{-1}^{0} \frac{1}{x^5} dx$ $\quad x=0$ is a discontinuity

$$\lim_{t \to 0^-} \int_{-1}^{t} x^{-5} dx$$

$$\lim_{t \to 0^-} \frac{x^{-4}}{-4} \Big|_{-1}^{t}$$

$$\lim_{t \to 0^-} \left[-\frac{1}{4t^4} - \left(-\frac{1}{4(-1)^4}\right)\right]$$

$$-\infty$$

$$\boxed{\text{Diverges}}$$

7. $\int_0^\infty e^{-x} dx$

$$\lim_{t \to \infty} \int_0^t e^{-x} dx$$

$$\lim_{t \to \infty} -e^{-x} \Big|_0^t$$

$$\lim_{t \to \infty} \left[-\frac{1}{e^t} - -\frac{1}{e^0}\right]$$

$$0 + \frac{1}{1}$$

$$\boxed{1}$$

8. Determine all the values of p for which $\int_0^1 \frac{1}{x^p} dx$ converges.

$$\lim_{t \to 0^+} \int_t^1 x^{-p} dx$$

$$\lim_{t \to 0^+} \left[\frac{x^{-p+1}}{-p+1}\right] \Big|_t^1$$

$$\lim_{t \to 0^+} \left[\frac{1}{-p+1} \cdot \frac{1}{x^{p-1}}\right] \Big|_t^1$$

$$\lim_{t \to 0^+} \left[\left(\frac{1}{-p+1} \cdot \frac{1}{1^{p-1}}\right) - \left(\frac{1}{-p+1} \cdot \frac{1}{t^{p-1}}\right)\right]$$

$$\left(\frac{1}{-p+1}\right) - \left(\frac{1}{-p+1} \cdot \frac{1}{t^{p-1}}\right)$$

For this to converge as $t \to 0^+$, $p-1$ must be negative. $\quad p-1<0$

$$\boxed{p<1}$$

9. If g is a twice-differentiable function, where $g(2) = 1$ and $\lim_{x\to\infty} g(x) = 8$, then $\int_2^\infty g'(x)\,dx$ is

$$\lim_{t\to\infty} \int_2^t g'(x)\,dx = \lim_{t\to\infty} [g(t) - g(2)]$$

$$8 - 1$$

A) -7 (B) 7 (C) 9 (D) nonexistent

---

10. If R is the unbounded region between the graph of $y = \frac{x}{(1+x^2)^2}$ and the x-axis for $x \geq 0$, then the area of R is

$$R = \lim_{t\to\infty} \int_0^t \frac{x}{(1+x^2)^2}\,dx$$

$u = 1+x^2$

$\frac{du}{2x} = dx$

$$R = \lim_{t\to\infty} \int_1^{1+t^2} \left(\tfrac{1}{2}u^{-2}\right) du$$

$$\lim_{t\to\infty} \left[-\frac{1}{2u}\right] \Big|_1^{1+t^2}$$

$$\lim_{t\to\infty} \left[\left(-\frac{1}{2+2t^2}\right) - \left(-\frac{1}{2}\right)\right]$$

$$0 + \frac{1}{2}$$

A) -1 (B) 0 (C) $\frac{1}{2}$ (D) infinite

---

11. For what values of p will $\int_1^\infty \frac{1}{x^{7p-3}}\,dx$ converge?

Converge if $7p-3 > 1$

$7p > 4$

$p > 4/7$

A) $p < 0$ (B) $p > 0$ (C) $p > \frac{4}{7}$ (D) $p < \frac{4}{7}$