7.7 Separation of Variables (Particular Solutions)
Calculus

Solutions

Practice

For each differential equation, find the solution that passes through the given initial condition.

1. $\frac{dy}{dx} = y\cos x$ and $y = 4$ when $x = 0$

$\frac{1}{y}dy = \cos x\, dx$

$\ln|y| = \sin x + C_1$

$|y| = e^{\sin x + C_1}$

$y = C_2 e^{\sin x}$

$4 = C_2 e^0$

$4 = C_2$

$\boxed{y = 4e^{\sin x}}$

2. $\frac{dy}{dx} = \frac{6x^2}{y}$ if $y(0) = -2$

$y\,dy = 6x^2\,dx$

$\frac{y^2}{2} = 2x^3 + C$

$\frac{(-2)^2}{2} = 2(0)^3 + C$

$2 = C$

$\frac{y^2}{2} = 2x^3 + 2$

$y^2 = 4x^3 + 4$

$y = \pm\sqrt{4x^3+4}$ y is negative

$\boxed{y = -\sqrt{4x^3+4}}$

3. $\frac{dy}{dx} = y\sin x$ if $y\left(\frac{\pi}{2}\right) = 2$

$\frac{1}{y}dy = \sin x\, dx$

$\ln|y| = -\cos x + C_1$

$|y| = \pm e^{-\cos x + C_1}$

$y = C_2 e^{-\cos x}$

$2 = C_2 e^0$

$2 = C_2$

$\boxed{y = 2e^{-\cos x}}$

4. $\frac{dy}{dx} = \frac{1}{5}(8 - y)$ and $y = 6$ when $x = 0$

$\frac{1}{8-y}dy = \frac{1}{5}dx$

$-\ln|8-y| = \frac{1}{5}x + C_1$

$\ln|8-y| = -\frac{1}{5}x + C_1$

$|8-y| = e^{-\frac{1}{5}x + C_1}$

$8 - y = C_2 e^{-\frac{1}{5}x}$

$y = C_2 e^{-\frac{1}{5}x} + 8$

$6 = C_2 e^0 + 8$

$-2 = C_2$

$\boxed{y = -2e^{-\frac{1}{5}x} + 8}$

5. $\frac{dy}{dx} = (y + 5)(x + 2)$ when $f(0) = 1$

$\frac{1}{y+5}dy = (x+2)dx$

$\ln|y+5| = \frac{x^2}{2} + 2x + C_1$

$|y+5| = C_2 e^{\frac{1}{2}x^2 + 2x}$

$1 + 5 = C_2 e^0$

$6 = C_2$

$|y+5| = 6e^{\frac{1}{2}x^2+2x}$

$\boxed{y = 6e^{\frac{1}{2}x^2+2x} - 5}$

6. $\frac{dy}{dx} = \frac{e^x}{y}$ if $y(0) = -4$

$y\,dy = e^x dx$

$\frac{1}{2}y^2 = e^x + C$

$\frac{1}{2}(16) = e^0 + C$

$8 = 1 + C$

$7 = C$

$\frac{1}{2}y^2 = e^x + 7$

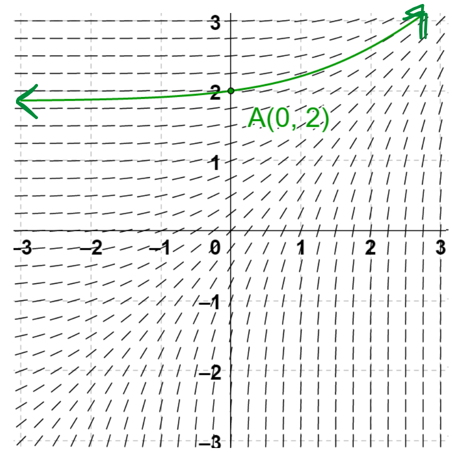
$y^2 = 2e^x + 14$

$y = \pm\sqrt{2e^x + 14}$

y is negative

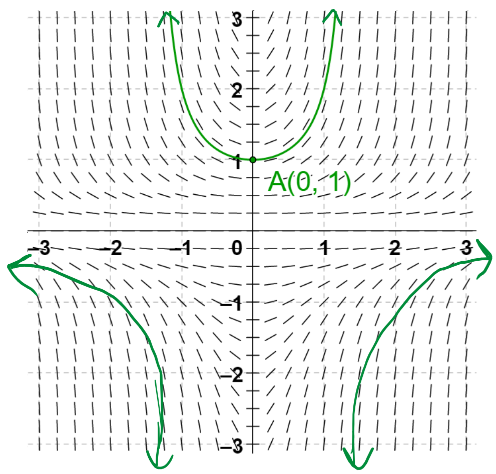
$\boxed{y = -\sqrt{2e^x + 14}}$

7. Find the particular solution to $\frac{dy}{dx} = e^{x-y}$ when $f(0) = 2$. Sketch the graph of this particular solution on the slope field provided.



$$\frac{dy}{dx} = \frac{e^x}{e^y}$$
$$e^y\,dy = e^x\,dx$$
$$e^y = e^x + C$$
$$e^2 = e^0 + C$$
$$e^2 - 1 = C$$
$$e^y = e^x + e^2 - 1$$
$$\boxed{y = \ln(e^x + e^2 - 1)}$$

8. Find the particular solution to $\frac{dy}{dx} = xy^2$ if $y = 1$ when $x = 0$. Sketch the graph of this particular solution on the slope field provided.



$$y^{-2}\,dy = x\,dx$$
$$-\frac{1}{y} = \frac{1}{2}x^2 + C$$
$$-1 = C$$
$$-\frac{1}{y} = \frac{1}{2}x^2 - 1$$
$$\frac{1}{y} = -\frac{1}{2}x^2 + 1$$
$$\boxed{y = \frac{1}{1 - \frac{x^2}{2}}}$$

7.7 Separation of Variables (Particular Solutions)

Test Prep

9. Consider the differential equation $\frac{dy}{dx} = e^y(4x - 1)$. Let $y = f(x)$ be the particular solution to the differential equation that passes through (2,0).

(a) Write an equation for the line tangent to the graph of f at the point (2,0). Use the tangent line to approximate $f(2.2)$.

$$\frac{dy}{dx} = e^0(8-1) = 7$$
$$\boxed{y = 7(x-2)}$$
$$y = 7(2.2-2)$$
$$y = 7(0.2)$$
$$\boxed{y = 1.4}$$

(b) Find $y = f(x)$, the particular solution to the differential equation that passes through (2,0).

$$e^{-y}\,dy = (4x-1)\,dx$$
$$-e^{-y} = 2x^2 - x + C$$
$$-e^0 = 8 - 2 + C$$
$$-1 = 6 + C$$
$$-7 = C$$
$$-e^{-y} = 2x^2 - x - 7$$
$$e^{-y} = -2x^2 + x + 7$$
$$-y = \ln(-2x^2 + x + 7)$$
$$\boxed{y = -\ln(-2x^2 + x + 7)}$$

10. One of Mr. Kelly's calculus students attempted to solve the differential equation $\frac{dy}{dx} = 2xy$ with initial condition $y = 3$ when $x = 0$. In which step, if any, does an error first appear?

Step 1: $\int \frac{1}{y} dy = \int 2x\, dx$
Step 2: $\ln|y| = x^2 + C$
Step 3: $|y| = e^{x^2} + C$ ← "plus C" should be in the exponent.
Step 4: Since $y = 3$ when $x = 0$, $3 = e^0 + C$.
Step 5: $y = -e^{x^2} + 2$

(A) Step 2 (B) Step 3 (C) Step 4 (D) Step 5 (E) There is no error in the solution.

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11. Consider the differential equation $\frac{dy}{dx} = 6 - 2y$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 4$.

a. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 0$. Use the tangent line to approximate $f(0.6)$.

$\frac{dy}{dx} = 6 - 2(4)$

$\frac{dy}{dx} = -2$

$\boxed{y - 4 = -2x}$

$y - 4 = -2(0.6)$

$y = -1.2 + 4$

$\boxed{y = 2.8}$

b. Find the value of $\frac{d^2y}{dx^2}$ at the point $(0, 4)$. Is the graph of $y = f(x)$ concave up or concave down at the point $(0, 4)$? Give a reason for your answer.

$\frac{d^2y}{dx^2} = -2\frac{dy}{dx}$

$\frac{d^2y}{dx^2} = -2(6 - 2y)$

$\left.\frac{d^2y}{dx^2}\right|_{(0,4)} = -2(6 - 2(4))$

$= -2(-2)$

$= 4$

$\boxed{\text{Concave up because } \frac{d^2y}{dx^2} > 0 \text{ at } (0,4)}$

c. Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(0) = 4$.

$\frac{1}{6-2y}\, dy = dx$

$-\frac{1}{2}\ln|6-2y| = x + C_1$

$\ln|6-2y| = -2x + C_2$

$|6-2y| = C_3 e^{-2x}$

$|6-2(4)| = C_3 e^0$

$2 = C$

$|6-2y| = 2e^{-2x}$

$6 - 2y = \pm 2e^{-2x}$

$-2y = \pm 2e^{-2x} - 6$

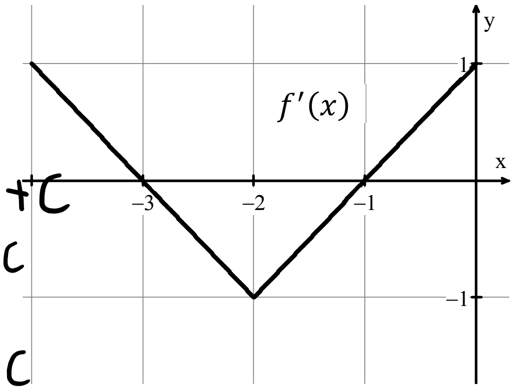
y is pos.

$\boxed{y = e^{-2x} + 3}$

d. For the particular solution $y = f(x)$ found in part c, find $\lim_{x\to\infty} f(x)$.

$\lim_{x\to\infty}\left(\frac{1}{e^{2x}} + 3\right) = 0 + 3 = \boxed{3}$

12. The graph of the derivative of f, f', is shown to the right. The domain of f is the set of all x such that $-4 < x < 0$.



Given that $f(-2) = 0$, find the solution $f(x)$.

If $-4<x<-2$, $f'(x) = -x-3 \rightarrow f(x) = -\frac{1}{2}x^2 - 3x + C$

$0 = -\frac{1}{2}(4) + 6 + C$

$-4 = C$

If $-2<x<0$, $f'(x) = x+1 \rightarrow f(x) = \frac{1}{2}x^2 + x + C$

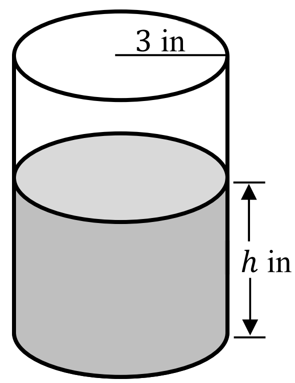
$0 = \frac{1}{2}(4) + (-2) + C$

$0 = C$

$$\boxed{f(x) = \begin{cases} -\frac{1}{2}x^2 - 3x - 4, & -4<x\leq -2 \\ \frac{1}{2}x^2 + x, & -2<x<0 \end{cases}}$$

The function exists at $x = -2$, so there should be an "or equal to" on one of the inequalities.

13. Mr. Bean's favorite addiction (rhymes with Poctor Depper) is put into a cylindrical container with radius 3 inches, as shown in the figure above. Let h be the depth of the soda in the container, measured in inches, where h is a function of time t, measured in minutes. The volume V of soda in the container is changing at the rate of $-\frac{\pi}{2}\sqrt{h}$ cubic inches per minute throughout the morning. Given that $h = 9$ at the start of 1st period ($t = 0$), solve the differential equation $\frac{dh}{dt}$ for h as a function of t. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)



$\frac{dV}{dt} = -\frac{\pi}{2}\sqrt{h}$

$V = 9\pi h$

$\frac{dV}{dt} = 9\pi \frac{dh}{dt}$

$-\frac{\pi}{2}\sqrt{h} = 9\pi \frac{dh}{dt}$

$-\frac{1}{18}\sqrt{h} = \frac{dh}{dt}$

$-\frac{1}{18}dt = h^{-\frac{1}{2}}dh$

$-\frac{1}{18}t = 2\sqrt{h} + C$

$-\frac{1}{18}(0) = 2\sqrt{9} + C$

$-6 = C$

$-\frac{1}{18}t = 2\sqrt{h} - 6$

$6 - \frac{1}{18}t = 2\sqrt{h}$

$3 - \frac{1}{36}t = \sqrt{h}$

$$\boxed{h = \left(3 - \frac{1}{36}t\right)^2}$$

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14. The rate at which a baby koala bear gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bear is first weighed, its weight is 2 pounds. If $B(t)$ is the weight of the bear, in pounds, at time t days after it is first weighed, then $\frac{dB}{dt} = \frac{1}{4}(20 - B)$. Find $y = B(t)$, the particular solution to the differential equation.

$\frac{1}{20-B}dB = \frac{1}{4}dt$

$-\ln|20-B| = \frac{1}{4}t + C_1$

$\ln|20-B| = -\frac{1}{4}t + C_2$

$|20-B| = C_3 e^{-\frac{1}{4}t}$

$|20-2| = C_3 e^0$

$18 = C_3$

$|20-B| = 18e^{-\frac{1}{4}t}$

$20 - B = \pm 18e^{-\frac{1}{4}t}$

$-B = \pm 18e^{-\,t} - 20$

$$\boxed{B(t) = -18e^{-\frac{1}{4}t} + 20}$$