7.9 Logistic Models

Calculus

Practice

The problems in this practice set are similar to what you would see on an AP Exam, so you will not have a "Test Prep" section.

1. A population y changes at a rate modeled by the logistic differential equation $\frac{dy}{dt} = 0.3y(4000 - y)$, where t is measured in years. What are all the values of y for which the population is increasing at a decreasing rate?

$y' > 0$ $\quad$ $y'' < 0$

$\frac{dy}{dt} = 1200y - 0.3y^2$

$\frac{d^2y}{dt^2} = 1200\frac{dy}{dt} - 0.6y\frac{dy}{dt}$

$0 = \frac{dy}{dt}(1200 - 0.6y)$

$y = 2000$

4,000 is the capacity limit.

y'': 0 $+$ 2000 $-$ 4000

$\boxed{2000 < y < 4000}$

2. A rumor spreads through a community at the rate $\frac{dy}{dt} = 2y(0.7 - y)$, where y is the proportion of the population that has heard the rumor at time t.

0.7 is the capacity limit

a. What proportion of the population has heard the rumor when it is $\underbrace{\text{spreading the fastest?}}_{\text{when } y' \text{ is a max, or } y'' \text{ changes sign.}}$

$y' = 1.4y - 2y^2$

$y'' = 1.4 - 4y$

$0 = 1.4 - 4y$

$-1.4 = -4y$

$\boxed{0.35 = y}$

b. If at $t = 0$, 20% of the people have heard the rumor, find y as a function of t.

Sep. of variables:

$\int \frac{1}{y(0.7-y)}\,dy = \int 2\,dt$

Linear partial fractions:

$\frac{A}{y} + \frac{B}{0.7-y} = \frac{1}{y(0.7-y)}$

$A(0.7-y) + By = 1$

$A(0.7) = 1 \qquad B(0.7) = 1$

$A = \frac{10}{7} \qquad B = \frac{10}{7}$

$\int \frac{10}{7y}\,dy + \int \frac{10}{7(0.7-y)}\,dy = \int 2\,dt$

$\frac{10}{7}\ln|y| + \underbrace{\frac{10}{7}\ln|0.7-y|}_{-1} = 2t + C$

$\ln\left|\frac{y}{0.7-y}\right| = \frac{7}{5}t + C_2$

$\frac{y}{0.7-y} = k e^{\frac{7}{5}t}$

[If $t=0$, $y=0.2$]

$\frac{0.2}{0.5} = k e^0 \rightarrow k = \frac{2}{5}$

$\left[\frac{y}{0.7-y} = \frac{2}{5}e^{\frac{7}{5}t}\right] 5(0.7-y)$

$5y = 1.4e^{\frac{7}{5}t} - 2ye^{\frac{7}{5}t}$

$5y + 2ye^{\frac{7}{5}t} = 1.4e^{\frac{7}{5}t}$

$y(5 + 2e^{\frac{7}{5}t}) = 1.4e^{\frac{7}{5}t}$

$\boxed{y = \frac{1.4e^{\frac{7}{5}t}}{5 + 2e^{\frac{7}{5}t}}}$

c. At what time t is the rumor spreading the fastest? [no calculator, give an exact answer.]

From part a, max at $y = 0.35$.

From part b, we know $\frac{y}{0.7-y} = \frac{2}{5}e^{\frac{7}{5}t}$

$\frac{0.35}{0.7-0.35} = \frac{2}{5}e^{\frac{7}{5}t}$

$1 = \frac{2}{5}e^{\frac{7}{5}t}$

$\frac{5}{2} = e^{\frac{7}{5}t}$

$\ln\frac{5}{2} = \frac{7}{5}t$

$\boxed{t = \frac{5}{7}\ln\frac{5}{2}}$

3. The population P of a city at time t is increasing according to a logistic differential equation. Which of the following could be the differential equation?

A. $\frac{dP}{dt} = 0.375t$ Quadratic function

B. $\frac{dP}{dt} = 0.375t(15000 - t)$ wrong variable

C. $\frac{dP}{dt} = 0.375P$ needs $(L - P)$

missing "P"

D. $\frac{dP}{dt} = 0.375(15000 - P)$

E. $\frac{dP}{dt} = 0.375P(15000 - P)$

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4. The total number of positive COVID cases in a city t days after the start of an outbreak is modeled by the function $y = C(t)$ that is the solution to the logistic differential equation $\frac{dy}{dt} = \frac{1}{7000}y(1600 - y)$. If there are 10 reported positive COVID cases initially, what is the limiting value for the total number of positive cases of the COVID virus as t increases?

The limiting value is the same as the "capacity limit". $\boxed{\text{1600 cases.}}$

The initial 10 cases is irrelevant.

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5. The size of a rabbit population is modeled by the function R that is a solution to the logistic differential equation $\frac{dR}{dt} = \frac{R}{3} - \frac{R^2}{2400}$, where t is measured in years for $t \geq 0$ and the initial population satisfies $R(0) > 0$. Which of the following statements could be true?

$\frac{1}{3}R\left(1 - \frac{R}{800}\right)$

✗ I. $\lim_{t\to\infty} R(t) > 1000$ no. $L = 800$

✓ II. The graph of R has a point of inflection for $t > 0$.

✓ III. The maximum rate of change of R occurs at $t = 0$.

Not guaranteed, but COULD be true. Don't confuse this with $R = \frac{L}{2}$ for a max rate.

A. None

B. II only

C. I & II only

D. II & III only

D

6. The rate of change $\frac{dP}{dt}$ of the number of people in a mall is modeled by a logistic differential equation. The maximum number of people allowed in the mall is 2000. At 10 A.M., the number of people in the mall is 200 and is increasing at a rate of 400 people per hour. Which of the following differential equations describe this situation?

We know the limiting value = 2000.

We know $\frac{dP}{dt} = 400$ when $P = 200$

A. $\frac{dP}{dt} = \frac{1}{400}(2000 - P) + 200$

B. $\frac{dP}{dt} = \frac{2}{5}(2000 - P)$

C. $\frac{dP}{dt} = \frac{1}{900}P(2000 - P)$

D. $\frac{dP}{dt} = 900P(2000 - P)$

E. $\frac{dP}{dt} = \frac{1}{400}P(2000 - P)$

C

$$\frac{dP}{dt} = \frac{k}{L}P(L - P)$$
$$400 = \frac{k}{2000}(200)(2000 - 200)$$
$$\frac{20}{9} = k$$
$$\frac{dP}{dt} = \frac{20}{9} \cdot \frac{1}{2000} P(2000 - P)$$

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7. The population P of deer in a preserve grows at a rate that is jointly proportional to the size of the deer population and the difference between the deer population and the carrying capacity of the population. If the carrying capacity of the preserve is 3000 deer, which of the following differential equations best models the growth rate of the deer population with respect to time t, where k is a constant?

A. $\frac{dP}{dt} = 3000k(1 - P)$

B. $\frac{dP}{dt} = 3000 - kP$

C. $\frac{dP}{dt} = k(3000 - P)$

D. $\frac{dP}{dt} = kP\left(1 - \frac{P}{3000}\right)$

E. $\frac{dP}{dt} = \frac{k}{P}(2000 - P)$

D

Size

$$\frac{dP}{dt} = S(L - P)$$
$$\frac{dP}{dt} = \frac{k}{3000}P(3000 - P)$$
$$\frac{dP}{dt} = kP\left(1 - \frac{P}{3000}\right)$$

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8. The rate of change, $\frac{dP}{dt}$, of the number of people entering an arena is modeled by a logistic differential equation. The capacity of the arena is 5000 people. At a certain time, the number of people in the arena is 1000 and is increasing at the rate of 500 people per minute. Which of the following differential equations could describe this situation?

A. $\frac{dP}{dt} = \frac{1}{800}(5000 - P)$

B. $\frac{dP}{dt} = \frac{1}{500}P(5000 - P)$

C. $\frac{dP}{dt} = \frac{1}{8000}P(5000 - P)$

D. $\frac{dP}{dt} = \frac{1}{5000}P(500 - P)$

C

$$\frac{dP}{dt} = \frac{k}{L}P(5000 - P)$$
$$500 = \frac{k}{5000}(1000)(5000 - 1000) = \frac{k}{5}(4000)$$
$$\frac{5}{8} = k$$
$$\frac{dP}{dt} = \frac{5}{8} \cdot \frac{1}{5000} P(5000 - P)$$

9. If a certain population is modeled by the function P that satisfies the logistic differential equation $\frac{dP}{dt} = 0.5P\left(1 - \frac{P}{200}\right)$, where t is the time in years and $P(0) = 100$. What is $\lim_{t\to\infty} P(t)$?

L

$\boxed{200}$

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10. The function P satisfies the logistic differential equation $\frac{dP}{dt} = \frac{P}{20}\left(1 - \frac{P}{1700}\right)$, where $P(0) = 210$. Which of the following statements is false?

$L = 1700$

A. $\lim_{t\to\infty} P(t) = 1700$ ✓

B. $\frac{dP}{dt}$ has a maximum value when $P = 210$. → should be $P = 850$.

C. $\frac{d^2P}{dt^2} = 0$ when $P = 850$ ✓ $\frac{1700}{2} = 850$

D. When $P > 850$, $\frac{dP}{dt} > 0$, $\frac{d^2P}{dt^2} < 0$ ✓

(?, 850)

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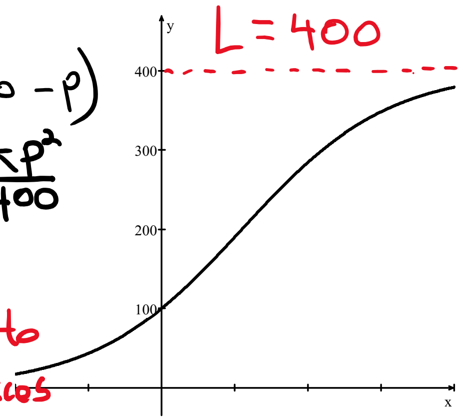
11. Which of the following differential equations for a population P could model the logistic growth shown in the figure?



$L = 400$

$$\frac{dP}{dt} = \frac{k}{400}P(400 - P)$$

$$\frac{dP}{dt} = kP - \frac{kP^2}{400}$$

Assume $k = 0.1$ to match answer choices

$$0.1P - \frac{0.1P^2}{400}$$

A. $\frac{dP}{dt} = 0.1P - 0.00025P^2$

B. $\frac{dP}{dt} = 0.1P - 0.025P^2$

✗ C. $\frac{dP}{dt} = 0.1P^2 - 0.00025P$

✗ D. $\frac{dP}{dt} = 0.1P + 0.00025P^2$

✗ E. $\frac{dP}{dt} = 0.1P + 0.025P^2$