9.1 Parametric Equations
Calculus

Solutions

Practice

1. For the given parametric equations, eliminate the parameter and write the corresponding rectangular equation. $x = e^{-t}$ and $y = e^{2t} - 1$.

$\ln x = -t$

$-\ln x = t$

$\ln\frac{1}{x} = t$

$x > 0$

$y = e^{2\ln\frac{1}{x}} - 1$

$y = e^{\ln\left(\frac{1}{x^2}\right)} - 1$

$\boxed{y = \frac{1}{x^2} - 1, \quad x > 0}$

2. Let C be a curve described by the parametrization $x = 5t$ and $y = t^4 + 3$. Find an expression for the slope of the line tangent to C at any point (x, y).

$\frac{dy}{dx} = \frac{4t^3}{5}$

3. The position of a particle at any time $t \geq 0$ is given by $x(t) = 3t^2 + 1$ and $y(t) = \frac{2}{3}t^3$. Find $\frac{dy}{dx}$ as a function of x.

$x - 1 = 3t^2$

$\frac{x-1}{3} = t^2$

$\sqrt{\frac{x-1}{3}} = t$

$\frac{dy}{dx} = \frac{2t^2}{6t} = \frac{1}{3}t$

$\boxed{\frac{dy}{dx} = \frac{1}{3}\sqrt{\frac{x-1}{3}}}$

4. A particle moves along the curve $xy + y = 9$. If $x = 2$ and $\frac{dy}{dt} = 3$, what is the value of $\frac{dx}{dt}$?

When $x = 2$, $2y + y = 9$

$3y = 9$

$y = 3$

$\frac{dx}{dt}y + x\frac{dy}{dt} + \frac{dy}{dt} = 0$

$\frac{dx}{dt}(3) + 2(3) + 3 = 0$

$3\frac{dx}{dt} + 9 = 0$

$\boxed{\frac{dx}{dt} = -3}$

5. A curve is described by the parametric equations $x = t\cos t$ and $y = t\sin t$. Find the equation of the line tangent to the curve at the point determined by $t = \pi$.

$x(\pi) = \pi\cos\pi = -\pi$
$y(\pi) = \pi\sin\pi = 0$

$\frac{dy}{dx} = \frac{\sin t + t\cos t}{\cos t + t(-\sin t)}$

$\frac{dy}{dx} = \frac{\sin\pi + \pi\cos\pi}{\cos\pi - \pi\sin\pi} = \frac{0+\pi(-1)}{-1-\pi(0)}$

$\frac{dy}{dx} = \pi$

$\boxed{y = \pi(x+\pi)}$

6. **Calculator active.** The coordinates $(x(t), y(t))$ of the position of a drone change at rates given by $x'(t) = 2t^3$ and $y'(t) = t^{\frac{1}{2}}$, where $x(t)$ and $y(t)$ are measured in meters and t is measured in seconds. At what time t, for $0 \le t \le 2$, does the slope of the line tangent to its path have a slope of 1.5?

$\frac{dy}{dx} = \frac{t^{1/2}}{2t^3}$

$1.5 = \frac{1}{2t^{5/2}}$

$3 = \frac{1}{t^{5/2}}$

$\frac{1}{3} = t^{5/2}$

$t = \left(\frac{1}{3}\right)^{2/5}$

$\boxed{t \approx 0.644}$

7. A curve in the xy-plane is defined by the parametric equations $x(t) = \cos(3t)$ and $y(t) = \sin(3t)$ for $t \ge 0$. What is the value of $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$?

$x'(t) = -3\sin(3t)$
$y'(t) = 3\cos(3t)$

$\sqrt{9\sin^2(3t) + 9\cos^2(3t)}$

$3\sqrt{\sin^2(3t) + \cos^2(3t)}$

$3\sqrt{1}$

$\boxed{3}$

8. A curve is defined by the parametric equations $x(t) = at^2 + b$ and $y(t) = ct - b$, where a, b, and c are nonzero constants. What is the slope of the line tangent to the curve at the point $(x(t), y(t))$ when $t = 2$?

$x(2) = 4a + b$
$y(2) = 2c - b$

$\frac{dy}{dx} = \frac{c}{2at}$

at $t=2 \rightarrow \frac{c}{2a(2)}$

$\boxed{\frac{c}{4a}}$

9. **No Calculator.** For $0 \le t \le 11$ the parametric equations $x = 3\sin t$ and $y = 2\cos t$ describe the elliptical path of an object. At the point where $t = 11$, the object travels along a line tangent to the path at that point. What is the slope of that line?

$\frac{dy}{dx} = \frac{-2\sin t}{3\cos t}$

$-\frac{2}{3}\tan t$

$\boxed{-\frac{2}{3}\tan(11)}$

10. A particle moves in the xy-plane so that its position for $t \ge 0$ is given by the parametric equations $x(t) = 2kt^2$ and $y(t) = 3t$, where k is a positive constant. When $t = 2$ the line tangent to the particle's path has a slope of 4. What is the value of k?

$\frac{dy}{dx} = \frac{3}{4kt} \longrightarrow 4 = \frac{3}{4k(2)}$

$32k = 3$

$\boxed{k = \frac{3}{32}}$

11. Find the equation of the line tangent to the curve defined parametrically by the equations $x(t) = t^3 + 2t$ and $y(t) = 2t^4 + 2t^2$ when $t = 1$.

$x(1) = 1+2=3 \qquad y(1)=2+2=4$

$\frac{dy}{dx} = \frac{8t^3+4t}{3t^2+2}$

at $t=1$, $\frac{dy}{dx} = \frac{8+4}{3+2} = \frac{12}{5}$

$\boxed{y-4 = \frac{12}{5}(x-3)}$

12. For what values of t does the curve given by the parametric equations $x(t) = \frac{1}{4}t^4 - \frac{9}{2}t^2$ and $y(t) = 3t^3 + 2t$ have a vertical tangent?

$\frac{dy}{dx} = \frac{9t^2+2}{t^3-9t}$

$t^3-9t=0$

$t(t^2-9)=0$

$\boxed{t=0,\ t=\pm 3}$

13. Suppose a curve is given by the parametric equations $x = f(t)$ and $y = g(t)$, for all $t > 1$ and $\frac{dy}{dt} = \frac{t^2+2}{t-1} * \frac{dx}{dt}$. What is the value of $\frac{dy}{dx}$ when $t = 2$?

$\frac{dy}{dx} = \frac{\frac{t^2+2}{t-1} \cdot \frac{dx}{dt}}{\frac{dx}{dt}} = \frac{t^2+2}{t-1}$

when $t=2$, $\frac{dy}{dx} = \frac{4+2}{2-1} = \boxed{6}$

9.1 Parametric Equations

Test Prep

14. A curve is defined parametrically by $x(t) = t^2$ and $y(t) = t^3 - 3t$. Find the points on the graph where the tangent line is horizontal or vertical.

$\frac{dy}{dx} = \frac{3t^2-3}{2t}$

Horizontal:

$3(t^2-1)=0$

$t^2=1$

$t=\pm 1$

$x(1)=1 \qquad y(1)=1-3=-2$

$x(-1)=1 \qquad y(-1)=-1+3=2$

$\boxed{(1,-2) \text{ and } (1,2)}$

Vertical:

$2t=0$

$t=0$

$x(0)=0 \qquad y(0)=0$

$\boxed{(0,0)}$

15. **Free Response**. Consider the curve given by the parametric equations $y = t^3 - 12t$ and $x = \frac{1}{2}t^2 - t$.

a. Find $\frac{dy}{dx}$ in terms of t.

$$\frac{dy}{dx} = \frac{3t^2 - 12}{t - 1}$$

b. Write an equation for the line tangent to the curve at the point where $t = -1$.

$x(-1) = \frac{1}{2} + 1 = \frac{3}{2}$

$y(-1) = -1 + 12 = 11$

$\frac{dy}{dx} = \frac{3-12}{-1-1} = \frac{-9}{-2}$

$$\boxed{y - 11 = \frac{9}{2}\left(x - \frac{3}{2}\right)}$$

c. Find the x and y coordinates for each $\underbrace{\text{critical point}}$ on the curve and identify each point as having a vertical or horizontal tangent.

where $\frac{dy}{dx} = 0$ or does not exist.

$3t^2 - 12 = 0$

$t^2 - 4 = 0$

$t = \pm 2$

$x(-2) = \frac{1}{2}(4) - (-2) = 4$ $\quad y(-2) = -8 + 24 = 16$

$x(2) = \frac{1}{2}(4) - 2 = 0$ $\quad y(2) = 8 - 24 = -16$

$t - 1 = 0$

$t = 1$

$x(1) = \frac{1}{2} - 1 = -\frac{1}{2}$

$y(1) = 1 - 12 = -11$

$$\boxed{\begin{array}{l}\text{Horizontal at } (4, 16) \text{ and } (0, -16) \\ \text{Vertical at } \left(-\frac{1}{2}, -11\right)\end{array}}$$

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16. A curve is given by the parametric equations $x(t) = 5t^3 - 5$ and $y(t) = t^2 + 7$. What is the equation of the tangent line to the curve when $t = 1$?

$x(1) = 5 - 5 = 0$

$y(1) = 1 + 7 = 8$

$\frac{dy}{dx} = \frac{2t}{15t^2} = \frac{2}{15}$

$y - 8 = \frac{2}{15}(x - 0)$

A. $x = 0$ B. $y = \frac{2}{15}x + 8$ C. $y = \frac{2}{15}x + 1$ D. $y = 8$ E. $y = \frac{15}{2}x + 7$