9.2 Second Derivatives of Parametric Equations
Calculus

Solutions

Practice

Given the following parametric equations, find $\frac{d^2y}{dx^2}$ in terms of t.

1. $x(t) = e^{-2t}$ and $y(t) = e^{2t}$.

$$\frac{dy}{dx} = \frac{2e^{2t}}{-2e^{-2t}} = -1 \cdot e^{2t} e^{2t} = -e^{4t}$$

$$\frac{d^2y}{dx^2} = \frac{-4e^{4t}}{-2e^{-2t}} = 2e^{4t}e^{2t}$$

$$\boxed{2e^{6t}}$$

2. $x(t) = t^3$ and $y(t) = t^4 + 1$ for $t > 0$.

$$\frac{dy}{dx} = \frac{4t^3}{3t^2} = \frac{4}{3}t$$

$$\frac{d^2y}{dx^2} = \frac{4/3}{3t^2} = \frac{4}{3} \cdot \frac{1}{3t^2}$$

$$\boxed{\frac{4}{9t^2}}$$

3. $x(t) = at^3$ and $y(t) = bt$, where a and b are positive constants.

$$\frac{dy}{dx} = \frac{b}{3at^2} = \frac{b}{3a}t^{-2}$$

$$\frac{d^2y}{dx^2} = \frac{-\frac{2b}{3a}t^{-3}}{3at^2}$$

$$= \boxed{-\frac{2b}{9a^2t^5}}$$

4. $\frac{dx}{dt} = 4$ and $\frac{dy}{dt} = \sin(t^2)$.

$$\frac{dy}{dx} = \frac{\sin t^2}{4} = \frac{1}{4}\sin t^2$$

$$\frac{d^2y}{dx^2} = \frac{\frac{1}{2}t \cdot \cos t^2}{4}$$

$$\boxed{\frac{t\cos t^2}{8}}$$

5. $x = e^t$ and $y = te^{-t}$.

$$\frac{dy}{dx} = \frac{(1)e^{-t} + (t)(-e^{-t})}{e^t} = \frac{e^{-t}(1-t)}{e^t}$$

$$= e^{-2t}(1-t)$$

$$\frac{d^2y}{dx^2} = \frac{-2e^{-2t}(1-t) + e^{-2t}(-1)}{e^t}$$

$$\frac{e^{-2t}[-2+2t-1]}{e^t} = \boxed{e^{-3t}(2t-3) \text{ or } \frac{2t-3}{e^{3t}}}$$

6. $x = t^2 + 1$ and $y = 2t^3$.

$$\frac{dy}{dx} = \frac{6t^2}{2t} = 3t$$

$$\frac{d^2y}{dx^2} = \boxed{\frac{3}{2t}}$$

7. Given a curve defined by the parametric equations $x(t) = 2 - t^2$ and $y(t) = t^2 + t^3$. Determine the open t-intervals on which the curve is concave up or down.

$$\frac{dy}{dx} = \frac{2t+3t^2}{-2t} = -1 - \frac{3}{2}t$$

$$\frac{d^2y}{dx^2} = \frac{-\frac{3}{2}}{-2t}$$

Concavity might change at $t=0$.

t	$(-\infty,0)$	0	$(0,\infty)$
$\frac{d^2y}{dx^2}$	neg.	DNE	pos.

$\boxed{\text{Concave down on } (-\infty,0). \text{ Concave up on } (0,\infty).}$

8. If $x(\theta) = 2 + \sec\theta$ and $y(\theta) = 1 + 2\tan\theta$, Find the slope and the concavity at $\theta = \frac{\pi}{6}$.

$$\frac{dy}{dx} = \frac{2\sec^2\theta}{\sec\theta\tan\theta} = \frac{2\sec\theta}{\tan\theta} = 2\frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}$$

$$\frac{dy}{dx} = \frac{2}{\sin\theta} \rightarrow \frac{2}{\sin\frac{\pi}{6}} = \frac{2}{\frac{1}{2}} = \boxed{4}$$

Slope

$$\frac{dy}{dx} = 2\csc\theta$$

$$\frac{d^2y}{dx^2} = \frac{-2\csc\theta\cot\theta}{\sec\theta\tan\theta} = -2\frac{1}{\sin}\frac{\cos}{\sin}\frac{\cos}{1}\frac{\cos}{\sin}$$

$$= -2\cot^3\theta$$

$$\left.\frac{d^2y}{dx^2}\right|_{\pi/6} = -2\left(\frac{\sqrt{3}}{1}\right)^3 = -6\sqrt{3}$$

$\boxed{\text{Concave down}}$

9. If $x = \cos\theta$ and $y = 3\sin\theta$, find the slope and concavity at $\theta = 0$.

$$\frac{dy}{dx} = \frac{3\cos\theta}{-\sin\theta} = -3\cot\theta$$

$$\left.\frac{dy}{dx}\right|_{\theta=0} = -3\cot(0) = \boxed{\text{undefined slope}}$$

$$\frac{d^2y}{dx^2} = \frac{3\csc^2\theta}{-\sin\theta} = -3\csc^3\theta$$

$$\left.\frac{d^2y}{dx^2}\right|_{\theta=0} = \text{undefined}$$

$\boxed{\text{Neither concave up or concave down.}}$

10. If $x(t) = t - \ln t$ and $y(t) = t + \ln t$, determine values of t where the graph is concave up.

$$\frac{dy}{dx} = \frac{1+\frac{1}{t}}{1-\frac{1}{t}}\cdot\frac{t}{t} = \frac{t+1}{t-1}$$

Simplify first!

$$\frac{d^2y}{dx^2} = \frac{\frac{(1)(t-1)-(t+1)(1)}{(t-1)^2}}{1-\frac{1}{t}} = \frac{\frac{-2}{(t-1)^2}}{\frac{t-1}{t}}$$

$t=0$, $t=1$ $\longrightarrow$ $$= \frac{-2t}{(t-1)^3}$$

t	$(-\infty,0)$	0	$(0,1)$	1	$(1,\infty)$
$\frac{d^2y}{dx^2}$	neg	0	pos	DNE	neg

$\boxed{\text{Concave up on } (0,1)}$

9.2 Second Derivatives of Parametric Equations

Test Prep

11. If $x = 3t^2 - 1$ and $y = \ln t$, what is $\frac{d^2y}{dx^2}$ in terms of t?

$$\frac{dy}{dx} = \frac{\frac{1}{t}}{6t} = \frac{1}{6t^2} = \frac{1}{6}t^{-2}$$

$$\frac{d^2y}{dx^2} = \frac{-\frac{1}{3}t^{-3}}{6t} = -\frac{1}{3}\frac{1}{t^3}\cdot\frac{1}{6t}$$

A. $\frac{1}{6}t^2$ B. $-\frac{1}{3}t^{-3}$ C. $-\frac{1}{18}t^{-4}$ D. $-\frac{1}{2}t^{-4}$ E. $6t^4$

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12. If $x = \theta - \cos\theta$ and $y = 1 - \sin\theta$, find the slope and concavity at $\theta = \pi$.

Slope: $\frac{dy}{dx} = \frac{-\cos\theta}{1+\sin\theta}\Big|_{\theta=\pi} = \frac{-(-1)}{1+(0)} = 1$

Concavity: $\frac{d^2y}{dx^2} = \frac{\frac{\sin\theta(1+\sin\theta) - -\cos\theta(\cos\theta)}{(1+\sin\theta)^2}}{1+\sin\theta}$

$\frac{d^2y}{dx^2}\Big|_{\theta=\pi} = \frac{\frac{0+1}{1^2}}{1} = 1$ up

A. Slope: -1, Concave down B. Slope: π, Concave up C. Slope: 1, Concave down

D. Slope: 1, Concave up E. Slope: $\frac{1}{\pi}$, Concave up