9.3 Arc Length (Parametric Form)
Calculus

Solutions

Practice

What is the length of the curve defined by the parametric equations? Solve without the use of a calculator.

1. $x(t) = 6t + 10$ and $y(t) = 14 - 4t$ for the interval $-1 \le t \le 3$?

$$\int_{-1}^{3} \sqrt{(6)^2 + (-4)^2}\, dt$$

$$\int_{-1}^{3} \sqrt{52}\, dt$$

$$t \cdot \sqrt{52} \Big|_{-1}^{3}$$

2.26
4.13

$$3\sqrt{52} - -1\sqrt{52}$$

$$\boxed{4\sqrt{52} \text{ or } 8\sqrt{13}}$$

2. $x = \frac{a}{2}t^2$ and $y = \frac{b}{2}t^2$, where a and b are constants. What is the length of the curve from $t = 0$ to $t = 1$?

$$\int_0^1 \sqrt{(at)^2 + (bt)^2}\, dt$$

$$\int_0^1 \sqrt{t^2(a^2+b^2)}\, dt = \int_0^1 t\sqrt{a^2+b^2}\, dt$$

$$\frac{t^2}{2}\sqrt{a^2+b^2} \Big|_0^1$$

$$\boxed{\frac{1}{2}\sqrt{a^2+b^2}}$$

3. $x(t) = 2t^2$ and $y(t) = \frac{2}{3}t^3$ for the interval $1 \le t \le 4$?

$$\int_1^4 \sqrt{(4t)^2 + (2t^2)^2}\, dt$$

$$\int_1^4 \sqrt{4t^2(4+t^2)}\, dt$$

$$\int_1^4 2t\sqrt{4+t^2}\, dt$$

$u = 4 + t^2$

$\frac{du}{2t} = dt$

$$\int_5^{20} \sqrt{u}\, du$$

$$\frac{u^{3/2}}{3/2} \Big|_5^{20} = \frac{2}{3}\left[(20)^{3/2} - (5)^{3/2}\right]$$

$$\frac{2}{3}\left[20\sqrt{20} - 5\sqrt{5}\right]$$

$$\frac{2}{3}\left[40\sqrt{5} - 5\sqrt{5}\right]$$

$$\boxed{\frac{70}{3}\sqrt{5}}$$

4. $x(\theta) = 5\cos\theta$ and $y(\theta) = 5\sin\theta$ for the interval $0 \le \theta \le 2\pi$.

$$\int_0^{2\pi} \sqrt{(-5\sin\theta)^2 + (5\cos\theta)^2}\, d\theta$$

$$\int_0^{2\pi} \sqrt{25(\sin^2\theta + \cos^2\theta)}\, d\theta$$

$$\int_0^{2\pi} 5\, d\theta$$

$$5\theta \Big|_0^{2\pi}$$

$$\boxed{10\pi}$$

5. $x(t) = 7t - 2$ and $y(t) = 4 - 8t$ for the interval $1 \le t \le 5$.

$$\int_1^5 \sqrt{(7)^2 + (-8)^2}\, dt$$

$$\int_1^5 \sqrt{113}\, dt$$

$$t\sqrt{113} \Big|_1^5$$

$$5\sqrt{113} - \sqrt{113}$$

$$\boxed{4\sqrt{113}}$$

6. If a curved is described by the parametric equations $x = t^2$ and $y = 2e^{2t}$, then which of the following gives the length of the path from $t = 0$ to $t = \ln 3$?

$x' = 2t$ $\quad$ $y' = 4e^{2t}$

$(x')^2 = 4t^2$ $\quad$ $(y')^2 = 16e^{4t}$

C

A. $\int_0^{\ln 3} \sqrt{4t^2 + 4e^{4t}}\, dt$

B. $\int_0^{\ln 3} \sqrt{t^4 + 4e^{4t}}\, dt$

C. $\int_0^{\ln 3} \sqrt{4t^2 + 16e^{4t}}\, dt$

D. $\int_0^{\ln 3} \sqrt{t^2 + 2e^{2t}}\, dt$

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7. Which of the following gives the length of the path described by the parametric equations $x = 2 + 4t$ and $y = 3 + t^2$ from $t = 0$ to $t = 1$?

$y' = 2t$ $\quad$ $x' = 4$

$(y')^2 = 4t^2$ $\quad$ $(x')^2 = 16$

D

A. $\int_0^1 \sqrt{4 + 2t}\, dt$

B. $\int_0^1 \sqrt{(2 + 4t)^2 + (3 + t^2)^2}\, dt$

C. $\int_0^1 \sqrt{16t^2 + t^4}\, dt$

D. $\int_0^1 \sqrt{16 + 4t^2}\, dt$

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8. Which of the following gives the length of the path described by the parametric equations $x = \cos t^3$ and $y = e^{5t}$ from $t = 0$ to $t = \pi$?

$y' = 5e^{5t}$ $\quad$ $x' = -3t^2 \sin t^3$

$(y')^2 = 25e^{10t}$ $\quad$ $(x')^2 = 9t^4 \sin^2 t^3$

A

A. $\int_0^{\pi} \sqrt{9t^4 \sin^2(t^3) + 25e^{10t}}\, dt$

B. $\int_0^{\pi} \sqrt{-3t^2 \sin(t^3) + 5e^{5t}}\, dt$

C. $\int_0^{\pi} \sqrt{9t^4 \sin^2(t^3) + 25e^{5t}}\, d$

D. $\int_0^{\pi} \sqrt{(\cos(t^3))^2 + (e^{5t})^2}\, dt$

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9. Which of the following gives the length of the path described by the parametric equations $x = \sin 3t$ and $y = \cos 2t$ from $t = 0$ to $t = \pi$?

$y' = -2\sin(2t)$ $\quad$ $x' = 3\cos(3t)$

$(y')^2 = 4\sin^2(2t)$ $\quad$ $(x')^2 = 9\cos^2(3t)$

D

A. $\int_0^{\pi} \sqrt{\sin^2 3t + \cos^2 2t}\, dt$

B. $\int_0^{\pi} \sqrt{\cos^2 3t + \sin^2 2t}\, dt$

C. $\int_0^{\pi} \sqrt{9\cos 3t + 4\sin 2t}\, dt$

D. $\int_0^{\pi} \sqrt{9\cos^2 3t + 4\sin^2 2t}\, dt$

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10. Which of the following gives the length of the path described by the parametric equations $x = \sqrt{t}$ and $y = 3t - 1$ from $0 \le t \le 1$?

$y' = 3$ $\quad$ $x' = \frac{1}{2}\frac{1}{\sqrt{t}}$

$(y')^2 = 9$ $\quad$ $(x')^2 = \frac{1}{4t}$

B

A. $\int_0^1 \sqrt{\frac{t}{4} + 9}\, dt$

B. $\int_0^1 \sqrt{\frac{1}{4}t^{-1} + 9}\, dt$

C. $\int_0^1 \sqrt{\frac{1}{4}t + 3}\, dt$

D. $\int_0^1 \sqrt{\frac{1}{2}t^{-\frac{1}{2}} + 3}\, dt$

No test prep. Problems 6-10 are great examples of problems you may see on the AP Exam.