9.5 Integrating Vector-Valued Functions

Calculus

Solutions

Practice

For problems 1-6, find the vector-valued function $f(t)$ that satisfies the given initial conditions.

1. $f(0) = \langle 2, 4\rangle$, $f'(t) = \langle 2e^t, 3e^{3t}\rangle$

$x = \int 2e^t$

$x = 2e^t + C$

$2 = 2 + C$

$0 = C$

$y = \int 3e^{3t}$ $\quad u = 3t$, $\frac{du}{3} = dt$

$y = \int 3e^u \frac{du}{3}$

$y = \int e^u du = e^{3t} + C$

$4 = e^0 + C$

$3 = C$

$$\boxed{f(t) = \langle 2e^t, e^{3t} + 3\rangle}$$

2. $f(0) = \langle \frac{1}{2}, -1\rangle$, $f'(t) = \langle te^{-t^2}, -e^{-t}\rangle$

$x = \int te^{-t^2} dt$ $\quad u = -t^2$, $\frac{du}{-2t} = dt$

$x = \int -\frac{1}{2} e^u du$

$x = -\frac{1}{2}e^{-t^2} + C$

$\frac{1}{2} = -\frac{1}{2} + C$

$1 = C$

$y = \int -e^{-t}$ $\quad u = -t$, $\frac{du}{-1} = dt$

$y = \int e^u du$

$y = e^{-t} + C$

$-1 = e^0 + C$

$-2 = C$

$$\boxed{f(t) = \langle -\frac{1}{2}e^{-t^2} + 1, e^{-t} - 2\rangle}$$

3. $f(0) = \langle 3, 1\rangle$, $f'(t) = \langle 6t^2, 4t\rangle$

$x = \int 6t^2 dt$ $\quad$ $y = \int 4t\, dt$

$x = 2t^3 + C$ $\quad$ $y = 2t^2 + C$

$3 = C$ $\quad$ $1 = C$

$f(t) = \langle 2t^3 + 3, 2t^2 + 1\rangle$

4. $f(0) = \langle -2, 5\rangle$, $f'(t) = \langle 2\cos t, -3\sin t\rangle$

$x = \int 2\cos t\, dt$ $\quad$ $y = \int -3\sin t\, dt$

$x = 2\sin t + C$ $\quad$ $y = 3\cos t + C$

$-2 = 2\sin 0 + C$ $\quad$ $5 = 3\cos(0) + C$

$-2 = C$ $\quad$ $5 = 3 + C$

$2 = C$

$f(t) = \langle 2\sin t - 2, 3\cos t + 2\rangle$

5. $f'(0) = \langle 3, 0\rangle, f(0) = \langle 0, 3\rangle$,
$f''(t) = \langle 5\cos t, -2\sin t\rangle$

$x' = \int 5\cos t\, dt$ $\quad$ $y' = \int -2\sin t\, dt$

$x' = 5\sin t + C$ $\quad$ $y' = 2\cos t + C$

$3 = C$ $\quad$ $0 = 2 + C$

$-2 = C$

$x = \int 5\sin t + 3\, dt$ $\quad$ $y = \int 2\cos t - 2\, dt$

$x = -5\cos t + 3t + C$ $\quad$ $y = 2\sin t - 2t + C$

$0 = -5 + 0 + C$ $\quad$ $3 = 2(0) - 2(0) + C$

$5 = C$ $\quad$ $3 = C$

$\boxed{f(t) = \langle -5\cos t + 3t + 5, 2\sin t - 2t + 3\rangle}$

6. $f'(0) = \langle 0, 2\rangle, f(0) = \langle 3, 0\rangle, f''(t) = \langle 4t^3, 3t^2\rangle$

$x' = \int 4t^3 dt$ $\quad$ $y' = \int 3t^2 dt$

$x' = t^4 + C$ $\quad$ $y' = t^3 + C$

$0 = 0 + C$ $\quad$ $2 = 0 + C$

$0 = C$ $\quad$ $2 = C$

$x = \int t^4 dt$ $\quad$ $y = \int t^3 + 2\, dt$

$x = \frac{1}{5}t^5 + C$ $\quad$ $y = \frac{1}{4}t^4 + 2t + C$

$3 = 0 + C$ $\quad$ $0 = 0 + 0 + C$

$\boxed{f(t) = \langle \frac{1}{5}t^5 + 3, \frac{1}{4}t^4 + 2t\rangle}$

7. **Calculator active.** For $t \geq 0$, a particle is moving along a curve so that its position at time t is $(x(t), y(t))$. At time $t = 1$, the particle is at position $(2, 4)$. It is known that $\frac{dx}{dt} = \frac{\sqrt{t+3}}{e^t}$ and $\frac{dy}{dt} = \cos^2 t$. Find the x-coordinate of the particles position at time $t = 5$.

$x(5) = 2 + \int_1^5 \frac{\sqrt{t+3}}{e^t} dt$

$\boxed{x(5) = 2.7988}$

8. The instantaneous rate of change of the vector-valued function $f(t)$ is given by $f'(t) = \langle 8t^3 + 2t, 10t^4\rangle$. If $f(1) = \langle 3, 7\rangle$, what is $f(-1)$?

$x(-1) = 3 + \int_1^{-1} 8t^3 + 2t\, dt$ $\quad$ $y(-1) = 7 + \int_1^{-1} 10t^4 dt$

$3 + [2t^4 + t^2] \Big|_1^{-1}$ $\quad$ $7 + [2t^5] \Big|_1^{-1}$

$3 + [2+1] - [2+1]$ $\quad$ $7 + [-2] - [2]$

$3 + 3 - 3$ $\quad$ $7 - 4$

3 $\quad$ 3

$\boxed{f(-1) = \langle 3, 3\rangle}$

9. **Calculator active.** At time $t \geq 0$, a particle moving in the xy-plane has velocity vector given by $v(t) = \langle 3t^2, 3 \rangle$. If the particle is at point $(1, 2)$ at time $t = 0$, how far is the particle from the origin at time $t = 2$?

$x = 1 + \int_0^2 3t^2\,dt \quad y = 2 + \int_0^2 3\,dt$

$x = 9 \quad y = 8$

distance from origin:

$\sqrt{9^2 + 8^2} \approx \boxed{12.0415}$

10. **Calculator active.** At time $t \geq 0$, a particle moving in the xy-plane has velocity vector given by $v(t) = \langle 2, \frac{\cos t}{e^t} \rangle$. If the particle is at point $(1, 2)$ at time $t = 0$, how far is the particle from the origin at time $t = 3$?

$x = 1 + \int_0^3 2\,dt \quad y = 2 + \int_0^3 \frac{\cos t}{e^t}\,dt$

$x = 7 \quad y \approx 2.528157388$

A

Distance from origin:

$\sqrt{7^2 + A^2} \approx \boxed{7.4425}$

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Test Prep

11. **Calculator active.** A remote controlled car travels on a flat surface. The car starts at the point with coordinates $(7, 6)$ at time $t = 0$. The coordinates $(x(t), y(t))$ of the position change at rates given by $x'(t) = -10 \sin t^2$ and $y'(t) = 9 \cos(2 + \sqrt{t})$, where $x(t)$ and $y(t)$ are measured in feet and t is measured in minutes. Find the y-coordinate of the position of the car at time $t = 1$.

$y(1) = 6 + \int_0^1 9\cos(2+\sqrt{t})\,dt$

$\boxed{y(1) \approx -1.789}$

12. The instantaneous rate of change of the vector-valued function $f(t)$ is given by $f'(t) = \langle 2 + 20t - 4t^3,\ 6t^2 + 2t \rangle$. If $f(1) = \langle 5, -3 \rangle$, what is $f(-1)$?

$x(-1) = 5 + \int_1^{-1} 2 + 20t - 4t^3\,dt$
$= 5 + \left[2t + 10t^2 - t^4\right]\Big|_1^{-1}$
$= 5 + [-2 + 10 - 1] - [2 + 10 - 1]$
$= 5 + 7 - 11$

$y(-1) = -3 + \int_1^{-1} 6t^2 + 2t\,dt$
$= -3 + \left[2t^3 + t^2\right]\Big|_1^{-1}$
$= -3 + [-2 + 1] - [2 + 1]$
$= -3 - 1 - 3$

$\boxed{f(-1) = \langle 1, -7 \rangle}$