9.7 Differentiating in Polar Form

Calculus

Solutions

Practice

Problems 1-5 are pre-calculus review on polar form.

1. Find the corresponding rectangular coordinates for the polar coordinates $\left(7, \frac{5\pi}{4}\right)$.

$x = 7\cos\left(\frac{5\pi}{4}\right)$ $y = 7\sin\left(\frac{5\pi}{4}\right)$

$x = 7\left(-\frac{\sqrt{2}}{2}\right)$ $y = 7\left(-\frac{\sqrt{2}}{2}\right)$

$\boxed{\left(-\frac{7\sqrt{2}}{2}, -\frac{7\sqrt{2}}{2}\right)}$

2. **Calculator active.** Find two sets of polar coordinates for the rectangular coordinate $(4, -2)$. Limit your answers on the interval $0 \leq \theta \leq 2\pi$.

$\tan\theta = -\frac{2}{4}$ $4^2 + (-2)^2 = r^2$

$\theta = -0.4636$ $20 = r^2$

$+2\pi$ $\sqrt{20} = r$

$\boxed{\left(\sqrt{20}, 5.819\right) \text{ or } \left(-\sqrt{20}, 2.678\right)}$

3. Convert the rectangular equation $\underbrace{x^2 + y^2}_{r^2} = 16$ to a polar equation.

$r^2 = 16$

$\boxed{r = 4}$

4. Convert the polar equation $r = 3\sec\theta$ to a rectangular equation.

$r = 3\frac{1}{\cos\theta}$

$r\cos\theta = 3$

$\boxed{x = 3}$

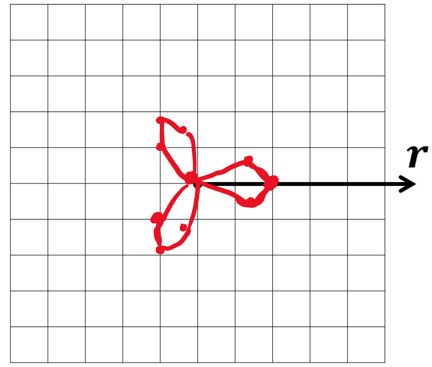
5. Sketch the polar curve $r = 2\cos 3\theta$ for $0 \le \theta \le \pi$ **without** a calculator, then check your answer.

Using a T-chart
Plug in values of θ.

r	θ
2	0
$\sqrt{2}$	$\pi/12$
0	$\pi/6$
$-\sqrt{2}$	$\pi/4$

r	θ
-2	$\pi/3$
$-\sqrt{2}$	$5\pi/12$
0	$\pi/2$

r	θ
$\sqrt{2}$	$7\pi/12$
2	$2\pi/3$
	$9\pi/12$



Find the slope of the line tangent to the polar curve at the given value of θ.

6. $r = 3\theta$ at $\theta = \frac{\pi}{2}$.

$x = 3\theta\cos\theta \qquad y = 3\theta\sin\theta$

$x' = 3\cos\theta - 3\theta\sin\theta \qquad y' = 3\sin\theta + 3\theta\cos\theta$

$r' = \dfrac{3\sin\theta + 3\theta\cos\theta}{3\cos\theta - 3\theta\sin\theta}$

$r'(\theta) = \dfrac{3(1) + 3(\frac{\pi}{2})(0)}{3(0) - 3(\frac{\pi}{2})(1)} = \dfrac{3}{-\frac{3\pi}{2}} = \boxed{-\dfrac{2}{\pi}}$

7. $r = \frac{5}{3-\cos\theta}$ at $\theta = \frac{3\pi}{2}$.

$x = \dfrac{5}{3-\cos\theta}\cdot\cos\theta \qquad y = \dfrac{5}{3-\cos\theta}\cdot\sin\theta$

$x' = \dfrac{-5\sin\theta[3-\cos\theta] - 5\cos\theta[\sin\theta]}{(3-\cos\theta)^2}$

$x'(\frac{3\pi}{2}) = \dfrac{-5(-1)(3-0) - 5(0)(-1)}{(3-0)^2} = 15/9$

$y' = \dfrac{5\cos\theta[3-\cos\theta] - 5\sin\theta[\sin\theta]}{(3-\cos\theta)^2}$

$y'(\frac{3\pi}{2}) = \dfrac{5(0)[3-0] - 5(-1)(-1)}{(3-0)^2} = -5/9$

$r'(\frac{3\pi}{2}) = \dfrac{y'(\frac{3\pi}{2})}{x'(\frac{3\pi}{2})} = \dfrac{-5/9}{15/9} = \boxed{-1/3}$

8. $r = \cos\theta$ at $\theta = \frac{\pi}{3}$.

$x = \cos\theta\cdot\cos\theta \qquad y = \cos\theta\cdot\sin\theta$

$x = \cos^2\theta$

$x' = -2\cos\theta\sin\theta \qquad y' = -\sin^2\theta + \cos^2\theta$

$x'(\frac{\pi}{3}) = -2(\frac{1}{2})(\frac{\sqrt{3}}{2}) = -\frac{\sqrt{3}}{2} \qquad y'(\frac{\pi}{3}) = -(\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2 = -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}$

$r'(\frac{\pi}{3}) = \dfrac{y'(\frac{\pi}{3})}{x'(\frac{\pi}{3})} = \dfrac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \boxed{\dfrac{1}{\sqrt{3}}}$

9. $r = 2(1 - \sin\theta)$ at $\theta = 0$.

$x = 2(1-\sin\theta)\cos\theta \qquad y = 2(1-\sin\theta)\sin\theta$

$x = 2\cos\theta - 2\sin\theta\cos\theta \qquad y = 2\sin\theta - 2\sin^2\theta$

$x' = -2\sin\theta - 2[\cos^2\theta + -\sin^2\theta]$

$x'(0) = -2(0) - 2[(1)^2 - (0)^2]$

$x'(0) = -2$

$y' = 2\cos\theta - 4\sin\theta\cos\theta$

$y'(0) = 2(1) - 4(0)(1) = 2$

$r'(0) = \dfrac{y'(0)}{x'(0)} = \dfrac{2}{-2} = \boxed{-1}$

10. A particle moves along the polar curve $r = 3\cos\theta$ so that $\frac{d\theta}{dt} = 2$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$. *Hint: remember implicit differentiation?*

$$\frac{dr}{dt} = -3\sin\theta \frac{d\theta}{dt}$$

At $\theta = \pi/3 \rightarrow -3\left(\frac{\sqrt{3}}{2}\right)(2)$

$$\boxed{-3\sqrt{3}}$$

11. A polar curve is given by the equation $r = \frac{15\theta}{\theta^2+1}$ for $\theta \geq 0$. What is the instantaneous rate of change of r with respect to θ when $\theta = 1$?

This problem is different! It is not the slope of the tangent line, it is instantaneous rate of change with respect to θ.

$$r'(\theta) = \frac{15(\theta^2+1) - 15\theta(2\theta)}{(\theta^2+1)^2}$$

$$r'(1) = \frac{15(2) - 30(1)(1)}{4} = \frac{0}{4} = \boxed{0}$$

12. Find the value(s) of θ where the polar graph $r = 2 - 2\cos\theta$ has a horizontal tangent line on the interval $0 \leq \theta \leq 2\pi$. Use a graphing calculator to verify your answers.

$r' = \frac{y'}{x'}$ Horizontal tangent when $y' = 0$.

$y = (2 - 2\cos\theta)\cdot\sin\theta$

$y' = 2\sin\theta\sin\theta + (2 - 2\cos\theta)\cos\theta$

$0 = 2\sin^2\theta + 2\cos\theta - 2\cos^2\theta$

$0 = 2(1 - \cos^2\theta) + 2\cos\theta - 2\cos^2\theta$

$0 = 2 - 4\cos^2\theta + 2\cos\theta$

$-2 \qquad -2$

$0 = 2\cos^2\theta - \cos\theta - 1$

$0 = (2\cos\theta + 1)(\cos\theta - 1)$

$\cos\theta = -\frac{1}{2}$ $\qquad$ $\cos\theta = 1$

$\boxed{\theta = \frac{2\pi}{3}, \frac{4\pi}{3}}$ $\qquad$ $\cancel{\theta = 0, 2\pi}$

13. Find the value(s) of θ where the polar graph $r = 3 - 3\sin\theta$ has a vertical tangent line on the interval $0 \leq \theta \leq 2\pi$. Use a graphing calculator to verify your answers.

Vertical tangent when $x' = 0$

$x = (3 - 3\sin\theta)\cos\theta$

$x' = (-3\cos\theta)\cos\theta + (3 - 3\sin\theta)(-\sin\theta)$

$0 = -3\cos^2\theta - 3\sin\theta + 3\sin^2\theta$

$0 = -3(1 - \sin^2\theta) - 3\sin\theta + 3\sin^2\theta$

$0 = -3 + 6\sin^2\theta - 3\sin\theta$

$3 \qquad 3$

$0 = 2\sin^2\theta - \sin\theta - 1$

$0 = (2\sin\theta + 1)(\sin\theta - 1)$

$\sin\theta = -\frac{1}{2}$ $\qquad$ $\sin\theta = 1$

$\boxed{\theta = 7\pi/6, 11\pi/6}$ $\qquad$ $\cancel{\theta = \pi/2, 3\pi/2}$

14. **Calculator active.** For a certain polar curve $r = f(\theta)$, it is known that $\frac{dx}{d\theta} = \cos\theta - \theta\sin\theta$ and $\frac{dy}{d\theta} = \sin\theta + \theta\cos\theta$. What is the value of $\frac{d^2y}{dx^2}$ at $\theta = 3$?

$$\frac{d^2y}{dx^2} = \left.\frac{\frac{d}{d\theta}\left[\frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}\right]}{\cos\theta - \theta\sin\theta}\right|_{\theta=3} \approx \frac{5.506731}{-1.413352} = \boxed{-3.896}$$

15. A polar curve is given by the differentiable function $r = f(\theta)$ for $0 \le \theta \le 2\pi$. If the line tangent to the polar curve at $\theta = \frac{\pi}{6}$ is vertical, which of the following must be true?

$$\text{Slope} = \frac{y'}{x'} = \frac{[f(\theta)\sin\theta]'}{[f(\theta)\cos\theta]' = 0} \rightarrow f'\left(\tfrac{\pi}{6}\right)\cos\tfrac{\pi}{6} + f\left(\tfrac{\pi}{6}\right)\cdot\left(-\sin\tfrac{\pi}{6}\right)$$

A. $f\left(\frac{\pi}{6}\right) = 0$ B. $f'\left(\frac{\pi}{6}\right) = 0$ C. $\frac{1}{2}f\left(\frac{\pi}{6}\right) - \frac{\sqrt{3}}{2}f'\left(\frac{\pi}{6}\right) = 0$ D. $\frac{\sqrt{3}}{2}f'\left(\frac{\pi}{6}\right) - \frac{1}{2}f\left(\frac{\pi}{6}\right) = 0$

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16. **Calculator active.** For $0 \le t \le 8$, a particle moving in the xy-plane has position vector $\langle x(t), y(t)\rangle = \langle \sin(2t), t^2 - t\rangle$, where $x(t)$ and $y(t)$ are measured in meters and t is measured in seconds.

a. Find the speed of the particle at time $t = 3$ seconds. Indicate units of measure.

$x' = 2\cos(2t)$

$y' = 2t - 1$

$$\sqrt{[2\cos(2(3))]^2 + (2(3)-1)^2}$$

$$\boxed{5.356}$$

b. At time $t = 5$ seconds, is the speed of the particle increasing or decreasing? Explain your answer.

$$\frac{d}{dt}\sqrt{[2\cos(2t)]^2 + (2t-1)^2}\,\Big|_{t=5} \approx 1.567$$

$$\boxed{\text{Increasing b/c } \tfrac{d}{dt}(\text{speed}) > 0.}$$

c. Find the total distance the particle travels over the time interval $0 \le t \le 6$ seconds.

$$\int_0^6 \sqrt{[2\cos(2t)]^2 + (2t-1)^2}\,dt \approx \boxed{32.436 \text{ meters}}$$

d. At time $t = 8$ seconds, the particle begins moving in a straight line. For $t \ge 8$, the particle travels with the same velocity vector that it had at time $t = 8$ seconds. Find the position of the particle at time $t = 11$ seconds.

Position at $t=8$ is $\langle x(8), y(8)\rangle = \langle -0.2879, 56\rangle$

Velocity at $t=8$ is $\langle x'(8), y'(8)\rangle = \langle -1.915318, 15\rangle$

Position at $t=11$ is $\langle x(8), y(8)\rangle + 3\cdot\langle x'(8), y'(8)\rangle$

$$\boxed{\langle -6.0338, 101\rangle}$$