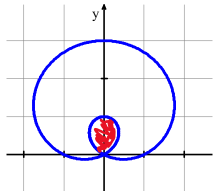
9.8 Area Bounded by a Polar Curve
Calculus

Solutions

Practice

Find the area of the given region for each polar curve.

1. Inside the smaller loop of the limaçon $r = 2\sin\theta + 1$.



$0 = 2\sin\theta + 1$

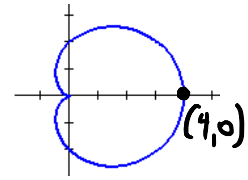
$-\frac{1}{2} = \sin\theta$

$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

$A = \frac{1}{2}\int_{7\pi/6}^{11\pi/6} [2\sin\theta + 1]^2 \, d\theta$

$\boxed{A \approx 0.5435}$

2. The region enclosed by the cardioid $r = 2 + 2\cos\theta$



$4 = 2 + 2\cos\theta$

$2 = 2\cos\theta$

$1 = \cos\theta$

$\theta = 0, 2\pi$

$A = \frac{1}{2}\int_0^{2\pi} [2 + 2\cos\theta]^2 \, d\theta$

$\boxed{A \approx 18.8495}$

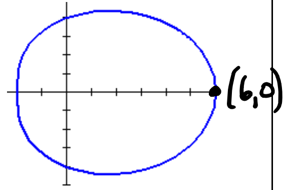
3. Inside the graph of the limaçon $r = 4 + 2\cos\theta$.

$6 = 4 + 2\cos\theta$

$2 = 2\cos\theta$

$1 = \cos\theta$

$\theta = 0, 2\pi$



$$A = \frac{1}{2}\int_0^{2\pi} [4 + 2\cos\theta]^2 \, d\theta$$

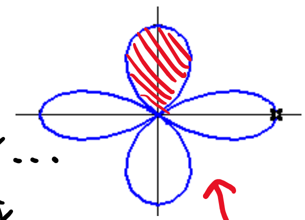
$$\boxed{A \simeq 56.5486}$$

4. Inside one petal of the four-petaled rose $r = \cos 2\theta$.

$0 = \cos 2\theta$

$2\theta = \pi/2, 3\pi/2, 5\pi/2, 7\pi/2 \ldots$

$\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4 \ldots$



One possible setup

$$A = \frac{1}{2}\int_{\pi/4}^{3\pi/4} [\cos(2\theta)]^2 \, d\theta$$

$$\boxed{A \simeq 0.3926}$$

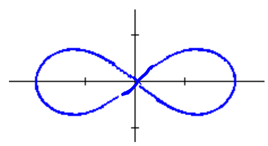
5. Inside one loop of the lemniscate $r^2 = 4\cos 2\theta$.

$0 = 4\cos 2\theta$

$0 = \cos 2\theta$

$2\theta = \pi/2, 3\pi/2, 5\pi/2, \ldots$

$\theta = \pi/4 \overset{neg}{\longmapsto} 3\pi/4 \overset{pos}{\longmapsto} 5\pi/4, \ldots$



$$A = \frac{1}{2}\int_{3\pi/4}^{5\pi/4} \left[\sqrt{4\cos(2\theta)}\right]^2 d\theta$$

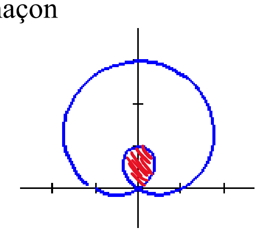
$$\boxed{A \simeq 2}$$

6. Inside the inner loop of the limaçon $r = 2\sin\theta - 1$.

$0 = 2\sin\theta - 1$

$\frac{1}{2} = \sin\theta$

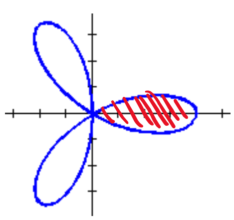
$\theta = \pi/6, 5\pi/6$



$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} [2\sin\theta - 1]^2 \, d\theta$$

$$\boxed{A \simeq 0.5435}$$

7. Write but do not solve, an expression that will give the area enclosed by one petal of the 3 petaled rose $r = 4\cos 3\theta$ found in the first and fourth quadrant.



$0 = 4\cos(3\theta)$

$0 = \cos(3\theta)$

$3\theta = \pi/2, 3\pi/2, 5\pi/2, \ldots$

$\theta = \pi/6, 3\pi/6, 5\pi/6, \ldots$

$$\boxed{\frac{1}{2}\int_{-\pi/6}^{\pi/6} [4\cos(3\theta)]^2 \, d\theta}$$

8. Write but do not solve an expression that can be used to find the area of the shaded region of the polar curve $r = 3 - 2\sin\theta$.

$1 = 3 - 2\sin\theta$

$1 = \sin\theta$

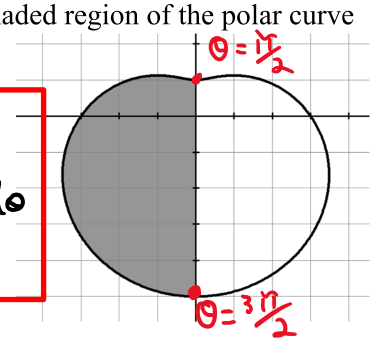
$\theta = \pi/2$

$5 = 3 - 2\sin\theta$

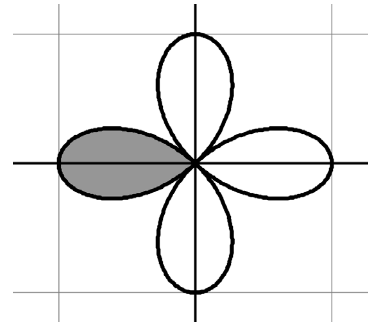
$-1 = \sin\theta$

$\theta = 3\pi/2$

$$\boxed{\frac{1}{2}\int_{\pi/2}^{3\pi/2} [3 - 2\sin\theta]^2 \, d\theta}$$



9. Write but do not solve an expression to find the area of the shaded region of the polar curve $r = \cos 2\theta$.



$0 = \cos(2\theta)$

$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \ldots$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \ldots$

$$\boxed{\frac{1}{2}\int_{3\pi/4}^{5\pi/4} [\cos(2\theta)]^2 \, d\theta}$$

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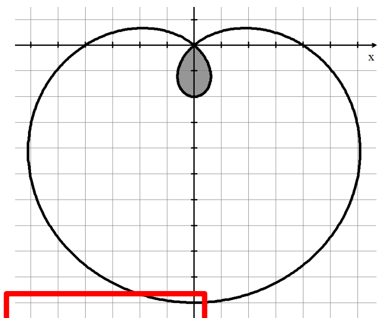
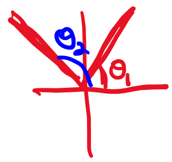
10. Find the area of the shaded region of the polar curve $r = 4 - 6\sin\theta$.



$0 = 4 - 6\sin\theta$

$\frac{2}{3} = \sin\theta$

$\sin^{-1}\left(\frac{2}{3}\right) = \theta$



$\theta_1 \approx 0.729727$

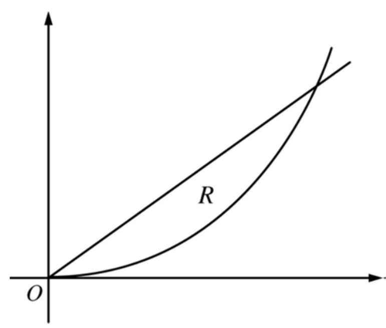
$\theta_2 = \pi - \theta_1 \approx 2.4118649$

$$\frac{1}{2}\int_{\theta_1}^{\theta_2} [4 - 6\sin\theta]^2 \, d\theta = \boxed{1.7635}$$

9.8 Area Bounded by a Polar Curve

Test Prep

11.



Let R be the region in the first quadrant that is bounded by the polar curves $r = \frac{\theta}{2}$ and $\theta = k$, where k is a constant, $0 < k < \frac{\pi}{2}$, as shown in the figure above. What is the area of R in terms of k?

$\frac{1}{2}\int_0^k \left(\frac{\theta}{2}\right)^2 d\theta$

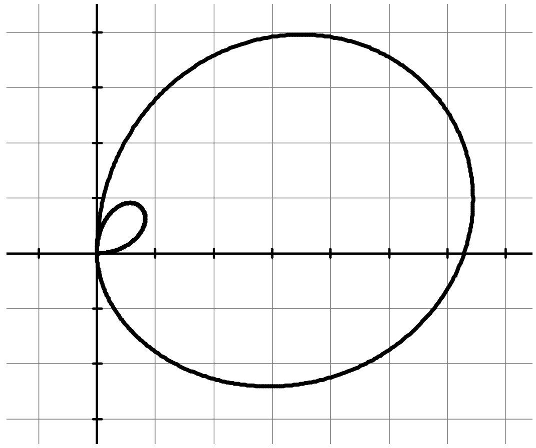
$\frac{1}{8}\int_0^k \theta^2 \, d\theta$

$\frac{1}{8} \cdot \frac{\theta^3}{3} \Big|_0^k$

$\frac{1}{24}\left[k^3 - 0\right]$

$$\boxed{\frac{k^3}{24}}$$

12.



Calculator active. Consider the polar curve defined by the function $r(\theta) = 2\theta\cos\theta$, where $0 \le \theta \le \frac{3\pi}{2}$. The derivative of r is given by $\frac{dr}{d\theta} = 2\cos\theta - 2\theta\sin\theta$. The figure above shows the graph of r for $0 \le \theta \le \frac{3\pi}{2}$.

a. Find the area of the region enclosed by the inner loop of the curve.

$2\theta\cos\theta = 0$

$\theta = 0$ or $\theta = \pi/2, 3\pi/2$

Checked by tracing the graph

$$A = \frac{1}{2}\int_0^{\pi/2} [2\theta\cos\theta]^2 \, d\theta$$

$$\boxed{A \approx 0.5065}$$

b. For $0 \le \theta \le \frac{3\pi}{2}$, find the greatest distance from any point on the graph of r to the origin. Justify your answer.

distance from origin $= r$

$r' = 2\cos\theta - 2\theta\sin\theta = 0$

$\theta \approx 0.860336 \to A$

$\theta \approx 3.4256195 \to B$

θ	r
0	0
A	1.122
B	-6.5767
$3\pi/2$	0

$\boxed{\text{Greatest distance is } 6.5767}$

c. There is a point on the curve at which the slope of the line tangent to the curve is $\frac{2}{2-\pi}$. At this point, $\frac{dy}{d\theta} = \frac{1}{2}$. Find $\frac{dx}{d\theta}$ at this point.

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{1/2}{x'(\theta)} = \frac{2}{2-\pi}$$

$$\frac{1}{2} \cdot \frac{2-\pi}{2} = x'(\theta)$$

$$\boxed{\frac{2-\pi}{4} = x'(\theta)}$$