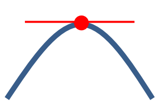
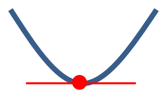
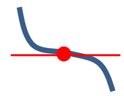
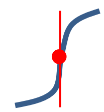
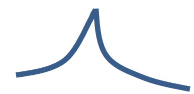
Unit 5 Review – Analytical Applications of Differentiation

This review summarizes everything from Unit 5 along with examples but contains no problems to work through.

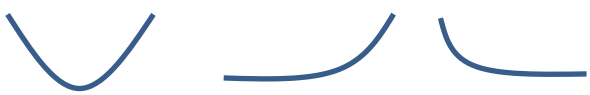
DEFINITIONS

Extrema: The maximum and minimum points. Extrema can be absolute or relative.

Critical Points: Where the first derivative is zero or DNE. These are possible maximum, minimum, or points of inflection!

$f'(x) = 0$	$f'(x) = 0$	$f'(x) = 0$	$f'(x) = DNE$	$f'(x) = DNE$
				
Horizontal Tangent Maximum	Horizontal Tangent Minimum	Horizontal Tangent Point of Inflection	Vertical Tangent Point of Inflection	Cusp (No Tangent) Maximum

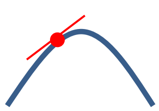
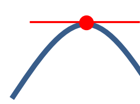
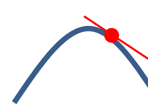
Concavity: Where the function is "cupping" up or down

CONCAVE UP	**CONCAVE DOWN**
	

Points of Inflection: Where the second derivative is zero or DNE and changes sign!

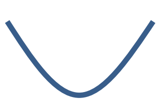
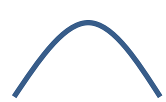
FIRST DERIVATIVE

The first derivative is the instantaneous rate of change, or the slope of the tangent line, and can determine if the function is increasing or decreasing at a given point.

$f'(x) > 0$	$f'(x) = 0$	$f'(x) < 0$
		
Function is increasing	Function is not increasing or decreasing	Function is decreasing

SECOND DERIVATIVE

The second derivative determines concavity.

$f''(x) > 0$	$f''(x) = 0$	$f''(x) < 0$
		
Concave Up	Neither concave up or concave down	Concave Down

FINDING EXTREMA

The First Derivative Test

STEPS	EXAMPLE $f(x) = x^2 + 2x + 1$
1. Find the critical points.	$f'(x) = 2x + 2$ $0 = 2x + 2$ $x = -1$
2. Determine whether the function is increasing or decreasing on each side of every critical point. A chart or number line helps!	<table><tr><th>Interval</th><th>$(-\infty, -1)$</th><th>-1</th><th>$(-1, \infty)$</th></tr><tr><td>Test Value</td><td>-2</td><td>-1</td><td>2</td></tr><tr><td>$f'(x)$</td><td>$f'(-2) = -$ Negative</td><td>$f'(-1)$ $= 0$</td><td>$f'(2) = 6$ Positive</td></tr></table>Function decreases to the left and increases to the right of $x = -1$ so it must be relative minimum point

The Second Derivative Test

STEPS	EXAMPLE $f(x) = x^2 + 2x + 1$
1. Find the critical points.	$f'(x) = 2x + 2$ $0 = 2x + 2$ $x = -1$
2. Determine whether the function is concave up or concave down at every critical point using the second derivative.	$f''(-1) = 2$ Second derivative is positive at $x = -1$ Concave up $x = -1$ is a relative minimum point

Finding Absolute Extrema on an Interval (Candidates Test)

STEPS	EXAMPLE $f(x) = x^2 + 2x + 1$ on the interval $[-3, 0]$
1. Find the critical points. The critical points are candidates as well as the endpoints of the interval.	$f'(x) = 2x + 2$ $0 = 2x + 2$ $x = -1$
2. Check all candidates using the $f(x)$.	$f(-3) = 4$ absolute maximum $f(-1) = 0$ absolute minimum $f(0) = 1$

LINEAR MOTION

The chart matches up function vocabulary with linear motion vocabulary.

FUNCTION	LINEAR MOTION
Value of a function at x	Position at time t
First Derivative	Velocity
Second Derivative	Acceleration
$f'(x) > 0$ Increasing Function	Moving right or up
$f'(x) < 0$ Decreasing Function	Moving left or down
$f'(x) = 0$	Not moving
Absolute Max	Farthest right or up
Absolute Min	Farthest left or down
$f'(x)$ changes signs	Object changes direction
$f'(x)$ and $f''(x)$ have same sign	Speeding Up
$f'(x)$ and $f''(x)$ have different signs	Slowing Down

Example:

A particle moves along the x-axis with the position function $x(t) = t^4 - 4t^3 + 2$ where $t > 0$.

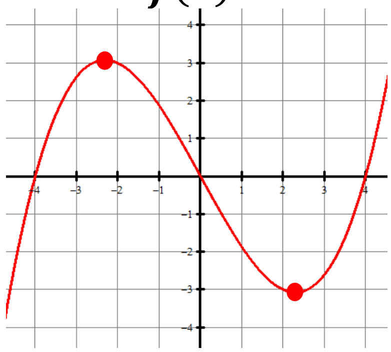
Interval	$(\mathbf{0}, \mathbf{2})$	**2**	$(\mathbf{2}, \mathbf{3})$	**3**	$(\mathbf{3}, \infty)$
$\boldsymbol{x'(t)}$ **velocity**	$x'(t) > 0$ increasing right	$x'(t) > 0$ Increasing right	$x'(t) > 0$ increasing right	$x'(t) = 0$ Not moving	$x'(t) < 0$ decreasing left
$\boldsymbol{x''(x)}$ **acceleration**	$x''(t) > 0$ Concave up	$x''(t) = 0$	$x''(t) < 0$ Concave down	$x''(t) < 0$ Concave down	$x''(t) < 0$ Concave down
Conclusion	Speeding Up	Moving Right	Slowing Down	Not Moving	Speeding Up

FUNCTION	LINEAR MOTION
$t = 3$ is maximum	$t = 3$ has no velocity Changing direction
Increasing (0,3)	Moving right (0,3)
Decreasing $(3, \infty)$	Moving left $(3, \infty)$

GRAPHICAL ANALYSIS

Connecting $f(x)$ to $f'(x)$

$f(x)$

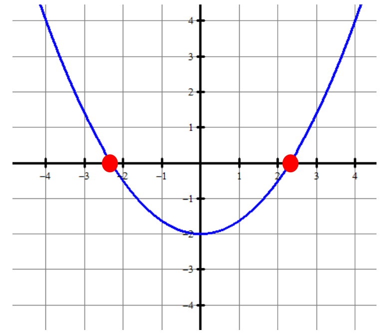


$f(x)$ max at $x = -2.2$ and min $x = 2.2$
so $f'(x) = 0$ at these points

$f(x)$ is increasing on
$(-\infty, -2.2)(2.2, \infty)$
so $f'(x) > 0$ on these intervals

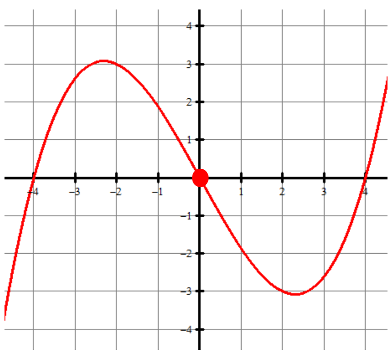
$f(x)$ is decreasing on $(-2.2, 2.2)$
so $f'(x) < 0$ on this interval

$f'(x)$



Connecting $f(x)$ to $f''(x)$

$f(x)$

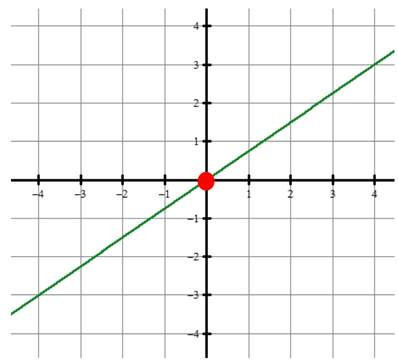


$f(x)$ is concave down on $(-\infty, 0)$
so $f''(x) < 0$ on this interval

$f(x)$ is concave up on $(0, \infty)$
so $f''(x) > 0$ on this interval

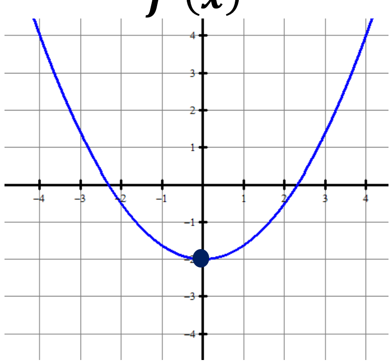
$x = 0$ is a point of inflection on $f(x)$
so $f''(x)$ changes sign at $x = 0$.

$f''(x)$



Connecting $f'(x)$ to $f''(x)$

$f'(x)$

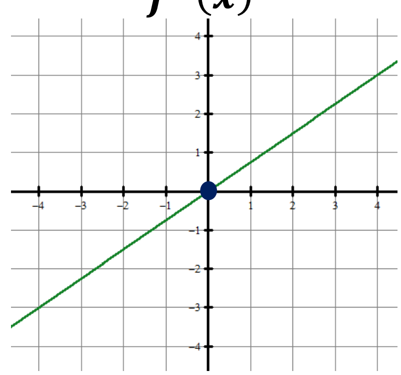


$f'(x)$ min $x = 0$
so $f''(x) = 0$ at $x = 0$

$f'(x)$ is decreasing on $(-\infty, 0)$
so $f''(x) < 0$ on this interval

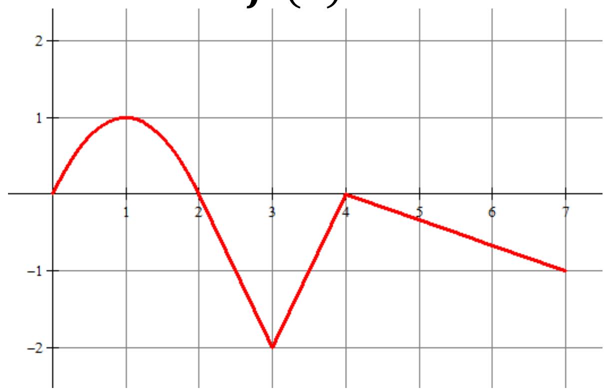
$f'(x)$ is increasing on $(0, \infty)$
so $f''(x) > 0$ on this interval

$f''(x)$



Using $f'(x)$ to draw conclusions about $f(x)$

$f'(x)$



Find Extrema of $f(x)$

$x = 2$ and 4 are critical points because $f'(x) = 0$

$x = 2$ is a maximum because $f'(x)$ is positive on left, negative on right

$x = 4$ is NOT an extrema because $f'(x)$ is negative on left, negative on right

Find Points of Inflection of $f(x)$

$x = 1, 3,$ and 4 are possible points of inflection because $f''(x) = 0$ or DNE

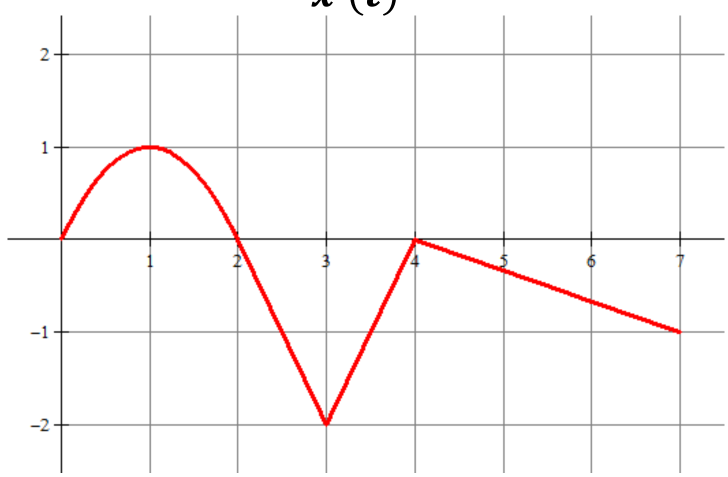
$x = 1$ is a point of inflection because $f''(x)$ changes sign from positive to negative at $x = 1$.

$x = 3$ is a point of inflection because $f''(x)$ changes sign from negative to positive at $x = 3$.

$x = 4$ is a point of inflection because $f''(x)$ changes sign from positive to negative at $x = 4$.

Now interpret the same graph as linear motion if the graph represents velocity of a particle moving along x-axis.

$x'(t)$



Moving Right or Left?

Particle moves right on (0,2).

$t = 2$ particle changes direction.

Particle moves left on (2,4)(4,7).

$t = 4$ particle has no velocity.

The maximum speed happens at $t = 3$.

Speeding up or Slowing down?

Particle speeds up on (0,1) because $f'(x)$ has the same sign as $f''(x)$

Particle slows down on (1,2) because $f'(x)$ has a different sign from $f''(x)$

Particle speeds up on (4,7) because $f'(x)$ has the same sign as $f''(x)$

Particle speeds up on (2,3) because $f'(x)$ has the same sign as $f''(x)$

Particle slows down on (3,4) because $f'(x)$ has a different sign from $f''(x)$