Name: ______________________________ Date:______________ Period: _____

Review

Unit 9 Review – Parametric Equations, Polar Coordinates, and Vector-Valued Functions

Reviews do NOT cover all material from the lessons but will hopefully remind you of key points. To be prepared, you must study all packets from Unit 9.

1. A curve is defined parametrically by $x(t) = t^3 - 3t^2 + 4$ and $y(t) = \sqrt{t^2 + 16}$. What is the equation of the tangent line at the point defined by $t = 3$?

$y(3) = \sqrt{25} = 5$

$x(3) = 27 - 27 + 4 = 4$

$y'(t) = \frac{2t}{2\sqrt{t^2+16}}$

$y'(3) = \frac{3}{5}$

$x'(t) = 3t^2 - 6t$

$x'(3) = 27 - 18 = 9$

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3/5}{9} = \frac{1}{15}$

$\boxed{y - 5 = \frac{1}{15}(x - 4)}$

2. An object moves in the xy-plane so that its position at any time t is given by the parametric equations $x(t) = t^2 + 3$ and $y(t) = t^3 + 5t$. What is the rate of change of y with respect to x when $t = 1$?

$y'(t) = 3t^2 + 5$

$y'(1) = 8$

$x'(t) = 2t$

$x'(1) = 2$

$\frac{dy}{dx}\Big|_{t=1} = \frac{y'(1)}{x'(1)} = \boxed{4}$

3. A curve in the xy-plane is defined by $(x(t), y(t))$, where $x(t) = 3t$ and $y(t) = t^2 + 1$ for $t \geq 0$. What is $\frac{d^2y}{dx^2}$ in terms of t?

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2t}{3}$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{2t}{3}\right]}{x'(t)} = \frac{2/3}{3} = \boxed{2/9}$

4. If $x(\theta) = \cot\theta$ and $y(\theta) = \csc\theta$, what is $\frac{d^2y}{dx^2}$ in terms of θ?

$\frac{dy}{dx} = \frac{-\csc\theta \cdot \cot\theta}{-\csc^2\theta} = \frac{\cot\theta}{\csc\theta} = \frac{\cos\theta}{\sin\theta} \cdot \sin\theta = \cos\theta$

$\frac{d^2y}{dx^2} = \frac{-\sin\theta}{-\csc^2\theta} = \boxed{\sin^3\theta}$

5. What is the length of the curve defined by the parametric equations $x(t) = 7 + 4t$ and $y(t) = 6 - t$ for the interval $0 \leq t \leq 9$?

$x' = 4$

$y' = -1$

$\int_0^9 \sqrt{(4)^2 + (-1)^2}\, dt$

$\int_0^9 \sqrt{17}\, dt$

$\sqrt{17}\, t \Big|_0^9 = \boxed{9\sqrt{17}}$

6. What is the length of the curve defined by the parametric equations $x(\theta) = 3\cos 2\theta$ and $y(\theta) = 3\sin 2\theta$ for the interval $0 \le \theta \le \frac{\pi}{2}$?

$x' = -6\sin 2\theta$

$y' = 6\cos 2\theta$

$\int_0^{\pi/2} \sqrt{36\sin^2(2\theta) + 36\cos^2(2\theta)}\, d\theta$

$\int_0^{\pi/2} \sqrt{36}\sqrt{\sin^2(2\theta) + \cos^2(2\theta)}\, d\theta$

$\int_0^{\pi/2} 6 \cdot \sqrt{1}\, d\theta$

$6\theta \Big|_0^{\pi/2}$

$\boxed{3\pi}$

7. If f is a vector-valued function defined by $\langle 2t^3 + 3t^2 + 4t + 1,\ t^3 - 4t - 1\rangle$ then $f''(2) =$

$f'(t) = \langle 6t^2 + 6t + 4,\ 3t^2 - 4\rangle$

$f''(t) = \langle 12t + 6,\ 6t\rangle$

$f''(2) = \boxed{\langle 30,\ 12\rangle}$

8. At time t, $0 \le t \le 2\pi$, the position of a particle moving along a path in the xy-plane is given by the vector-valued function, $f(t) = \langle e^t \sin 3t,\ e^t \cos 3t\rangle$. Find the slope of the path of the particle at time $t = \frac{\pi}{6}$.

$$f'(t) = \frac{y'}{x'} = \frac{e^t\cos 3t - 3e^t \sin 3t}{e^t \sin 3t + 3e^t \cos 3t} = \frac{\cos(3t) - 3\sin(3t)}{\sin(3t) + 3\cos(3t)}$$

$$f'(\pi/6) = \frac{\cos(\pi/2) - 3\sin(\pi/2)}{\sin(\pi/2) + 3\cos(\pi/2)} = \frac{0-3}{1+0} = \boxed{-3}$$

9. Find the vector-valued function $f(t)$ that satisfies the initial conditions $f(0) = \langle -2, 5\rangle$ and $f'(t) = \langle 10t^4,\ 2t\rangle$.

$\int 10t^4\, dt$ | $\int 2t\, dt$

$2t^5 + C$ | $t^2 + C$

$-2 = 2(0)^5 + C$ | $5 = (0)^2 + C$

$-2 = C$ | $5 = C$

$\boxed{f(t) = \langle 2t^5 - 2,\ t^2 + 5\rangle}$

10. **Calculator active**: For $t \ge 0$, a particle is moving along a curve so that its position at time t is $(x(t), y(t))$. At time $t = 1$ the particle is at position $(3, 4)$. It is known that $\frac{dx}{dt} = \sin 2t$ and $\frac{dy}{dt} = \frac{\sqrt{t}}{e^{2t}}$. Find the y-coordinate of the particles position at time $t = 3$.

$y(1) + \int_1^3 \frac{dy}{dt}\, dt$

$4 + \int_1^3 \frac{\sqrt{t}}{e^{2t}}\, dt \approx \boxed{4.0796}$

11. A particle moving in the xy-plane has position given by parametric equations $x(t) = t$ and $y(t) = 4 - t^2$.

A. Find the velocity vector.

$$\boxed{\langle 1, -2t \rangle}$$

B. Find the speed when $t = 1$.

$$\sqrt{(1)^2 + (-2(1))^2} = \boxed{\sqrt{5}}$$

C. Find the acceleration vector.

$$\boxed{\langle 0, -2 \rangle}$$

12. It is known the acceleration vector for a particle moving in the xy-plane is given by $a(t) = \langle t, \sin t \rangle$. When $t = 0$, the velocity vector $v(0) = \langle 0, -1 \rangle$ and the position vector $p(0) = \langle 0, 0 \rangle$. Find the position vector at time $t = 2$.

x': $\int t \, dt$, $\frac{1}{2}t^2 + C$, $0 = \frac{1}{2}(0)^2 + C$, $0 = C$

y': $\int \sin t \, dt$, $-\cos t + C$, $-1 = -\cos(0) + C$, $-1 = -1 + C$, $0 = C$

x: $\int \frac{1}{2}t^2 \, dt$, $\frac{1}{6}t^3 + C$, $0 = \frac{1}{6}(0)^3 + C$, $0 = C$

y: $\int -\cos t \, dt$, $-\sin t + C$, $0 = -\sin(0) + C$, $0 = C$

$p(t) = \langle \frac{1}{6}t^3, -\sin t \rangle$

$$\boxed{p(2) = \langle \tfrac{4}{3}, -\sin 2 \rangle}$$

13. Find the slope of the tangent line to the polar curve $r = 2\cos 4\theta$ at the point where $\theta = \frac{\pi}{4}$.

$x(\theta) = r\cos\theta$

$x(\theta) = 2\cos 4\theta \cdot \cos\theta$

$x'(\theta) = -8\sin 4\theta \cos\theta - 2\cos 4\theta \sin\theta$

$x'(\frac{\pi}{4}) = -8\sin(\pi)\cos(\frac{\pi}{4}) - 2\cos(\pi)\sin\frac{\pi}{4}$

$x'(\frac{\pi}{4}) = 0 - 2(-1)(\frac{\sqrt{2}}{2}) = \sqrt{2}$

$y(\theta) = r\sin\theta$

$y(\theta) = 2\cos 4\theta \cdot \sin\theta$

$y'(\theta) = -8\sin 4\theta \sin\theta + 2\cos 4\theta \cos\theta$

$y'(\frac{\pi}{4}) = -8\sin(\pi)\sin(\frac{\pi}{4}) + 2\cos(\pi)\cos(\frac{\pi}{4})$

$y'(\frac{\pi}{4}) = 0 + 2(-1)(\frac{\sqrt{2}}{2}) = -\sqrt{2}$

$$\text{Slope} = \frac{y'(\theta)}{x'(\theta)} = \frac{-\sqrt{2}}{\sqrt{2}} = \boxed{-1}$$

14. **Calculator active.** For a certain polar curve $r = f(\theta)$, it is known that $\frac{dx}{d\theta} = \cos\theta - \theta\sin\theta$ and $\frac{dy}{d\theta} = \sin\theta + \theta\cos\theta$. What is the value of $\frac{d^2y}{dx^2}$ at $\theta = 6$?

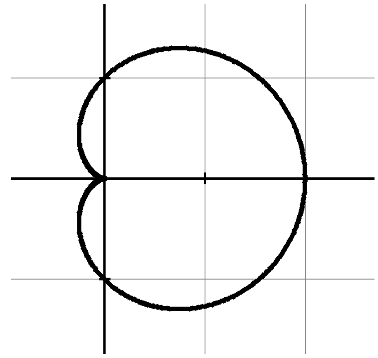
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left[\frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}\right]}{\cos\theta - \theta\sin\theta} \text{ at } \theta = 6 \rightarrow \frac{5.466085003}{2.636663276}$$

$$\boxed{2.073}$$

15. **Calculator active.** Find the total area enclosed by the polar curve $r = 1 + \cos\theta$ shown in the figure above.

$$\int_0^{2\pi} \tfrac{1}{2}\left[1+\cos\theta\right]^2 d\theta$$

$$\boxed{4.712}$$



16. **Calculator active.** Find the area of the inner loop of the polar curve $r = 3 - 6\sin\theta$.

$0 = 3 - 6\sin\theta$

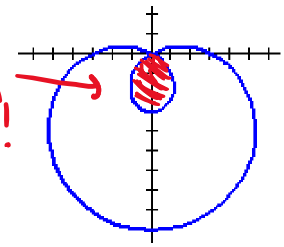
$\tfrac{1}{2} = \sin\theta$

$\theta = \tfrac{\pi}{6}, \tfrac{5\pi}{6}$

$$\int_{\pi/6}^{5\pi/6} \tfrac{1}{2}\left[3-6\sin\theta\right]^2 d\theta$$

$$\boxed{4.8916}$$

Graph it!



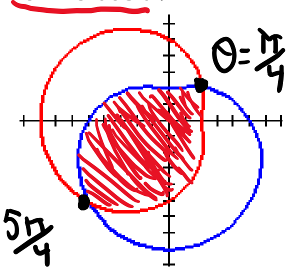
17. Find the total area of the common interior of the polar graphs $r = 5 - 3\sin\theta$ and $r = 5 - 3\cos\theta$.

$5 - 3\sin\theta = 5 - 3\cos\theta$

$\sin\theta = \cos\theta$

$\theta = \tfrac{\pi}{4}, \tfrac{5\pi}{4}$

Graph it!



$$\int_{-3\pi/4}^{\pi/4} \tfrac{1}{2}\left[5-3\cos\theta\right]^2 d\theta + \int_{\pi/4}^{5\pi/4} \tfrac{1}{2}\left[5-3\sin\theta\right]^2 d\theta \qquad -\tfrac{3\pi}{4} = \tfrac{5\pi}{4}$$

$$\boxed{50.2505}$$

18. **Calculator active.** The figure shows the graphs of the polar curves $r = 4\cos 3\theta$ and $r = 4$. What is the sum of the areas of the shaded regions?

Careful!

$$\underbrace{\int_0^{2\pi} \tfrac{1}{2}\left[4\right]^2 d\theta}_{\text{Circle}} - \underbrace{\int_0^{\pi} \tfrac{1}{2}\left[4\cos(3\theta)\right]^2 d\theta}_{\text{rose curve}}$$

$$\boxed{37.699}$$